15.9. **Change of variables in multiple integrals.** In the following, we consider the change of variable in multiple integrals. Note that, for multi-variables domains, the change of variable is a transformation. Typical examples consist of changing the two-dimensional Cartesian coordinate into the polar coordinate and changing the three-dimensional Cartesian coordinate into the cylindrical or spherical coordinates. Generally, a transformation T is a mapping from the uv-plane into the xy-plane, say $(x,y) = T(u,v)$. T is called one-to-one if no two points have the same image. When T is one-to-one, we write T^{-1} for the inverse transformation of T that maps points from the xy-plane into the uv-plane.

Example 15.26. Let $(x,y) = T(u,v)$ be a transformation given by $x = u^2 - v^2$ and $y = 2uv$. Consider the square $S = \{(u,v)|0 \le u \le 1,\ 0 \le v \le 1\}$. In some computations, one can see that

$$\begin{cases} T(\{(u,0)|0 \le u \le 1\}) = \{(x,0)|0 \le x \le 1\}, \\ T(\{(0,v)|0 \le v \le 1\}) = \{(x,0)| -1 \le x \le 0\}, \\ T(\{(u,1)|0 \le u \le 1\}) = \{(x,y)|x = y^2/4 - 1,\ 0 \le y \le 2\}, \\ T(\{(1,v)|0 \le v \le 1\}) = \{(x,y)|x = 1 - y^2/4,\ 0 \le y \le 2\}. \end{cases}$$

To see how the change of variable works for double integrals, let S be the square with corners (u_0,v_0), $(u_0+\Delta u, v_0)$, $(u_0, v_0+\Delta v)$, $(u_0+\Delta u, v_0+\Delta v)$ and let $T(S) = R$. For convenience, we write $T = (g,h)$ and let $(x_0,y_0) = T(u_0,v_0)$. In this setting, the area $A(R)$ of R is approximately

$$A(R) \approx |T(u_0+\Delta u, v_0) - T(u_0,v_0)| \times |T(u_0, v_0+\Delta v) - T(u_0,v_0)| \sin\theta,$$

where θ is the angle between vectors $T(u_0+\Delta u, v_0) - T(u_0,v_0)$ and $T(u_0,v_0+\Delta v) - T(u_0,v_0)$. Assume that g,h are continuously differentiable. By the mean value theorem, one has

$$T(u_0+\Delta u, v_0) - T(u_0,v_0) \approx \Delta u \langle g_u(u_0,v_0), h_u(u_0,v_0)\rangle$$

and

$$T(u_0, v_0+\Delta v) - T(u_0,v_0) \approx \Delta v \langle g_v(u_0,v_0), h_v(u_0,v_0)\rangle.$$

Consequently, this leads to

$$A(R) \approx \text{abs}\left(\begin{vmatrix} g_u(u_0,v_0) & g_v(u_0,v_0) \\ h_u(u_0,v_0) & h_v(u_0,v_0) \end{vmatrix} \right) A(S),$$

where $A(S)$ is the area of S.

Definition 15.4. The Jacobian of a transformation $(x,y) = (g(u,v), h(u,v))$ is defined by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = g_u h_v - g_v h_u.$$

Consider the case that S is the rectangle $[a,b]\times[c,d]$ in the uv-plane. Set $\Delta u = (b-a)/m$, $\Delta v = (d-c)/n$, $u_i = a + i\Delta x$, $v_j = c + j\Delta v$ and $S_{ij} = [u_{i-1},u_i]\times[v_{j-1},v_j]$. Select $(u_{ij}^*, v_{ij}^*) \in S_{ij}$ and define correspondingly $(x_{ij}^*, y_{ij}^*) = T(u_{ij}^*, v_{ij}^*)$, $R_{ij} = T(S_{ij})$ and $R = T(S)$. Let $J = g_u h_v - g_v h_u$. When g,h are continuously differentiable, $A(R_{ij}) \approx |J(u_i,v_j)|A(S_{ij}) \approx |J(u_{ij}^*, v_{ij}^*)|A(S_{ij})$, one has

$$\begin{aligned} \iint_R f(x,y)dA &\approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)A(R_{ij}) \approx \sum_{i=1}^m \sum_{j=1}^n f(T(u_{ij}^*, v_{ij}^*))|J(u_{ij}^*, v_{ij}^*)|A(S_{ij}) \\ &\approx \iint_S f(T(u,v))|J(u,v)|dA, \end{aligned}$$

Theorem 15.5 (Change of variables in double integrals). *Let $(x, y) = T(u, v)$ be a continuously differentiable transformation with domain S. Assume that T is one-to-one on S except on the boundary of S and the Jacobian of T is non-zero on S. Assume further that S and $T(S)$ are of types I or II. Then, for any function f continuous on $T(S)$,*

$$\iint_{T(S)} f(x,y)dA = \iint_S f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA.$$

Example 15.27. Let R be the region bounded by parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$ and above the x-axis. By using the change of variables $x = u^2 - v^2$ and $y = 2uv$, one may check that R is the image of $S = \{(u,v)|0 \le u \le 1, 0 \le v \le 1\}$ under this transformation. Note that $\frac{\partial(x,y)}{\partial(u,v)} = 4(u^2 + v^2)$. By Theorem 15.5, we obtain

$$\iint_R y dA = \iint_S 2uv \cdot 4(u^2 + v^2) dA = 2.$$

Example 15.28. Consider the integral $\iint_R e^{(x+y)/(x-y)} dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$. Let $x = u + v$, $y = u - v$ and $S = \{(u,v)|1/2 \le v \le 1, -v \le u \le v\}$. It is easy to check that R is the image of S under this transformation. As $\frac{\partial(x,y)}{\partial(u,v)} = -2$, we obtain

$$\iint_R e^{(x+y)/(x-y)} dA = 2 \iint_S e^{-u/v} dA = 2 \int_{1/2}^1 \int_{-v}^v e^{-u/v} du dv = \frac{3(e - 1/e)}{4}.$$

Example 15.29. Consider the integral $\iint_R xy dA$, where R is the region in the first quadrant bounded by the lines $y = x$, $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$. Through the change of variables, $x = u/v$ and $y = v$, one can see that R is the image of $S = \{(u,v)|1 \le u \le 3, \sqrt{u} \le v \le \sqrt{3u}\}$. Note that $\frac{\partial(x,y)}{\partial(u,v)} = 1/v$. This implies

$$\iint_R xy dA = \iint_S u/v dA = \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u/v dv du = \frac{\ln 3}{2} \int_1^3 u du = 2 \ln 3.$$

For triple integrals, consider the change of variables $(x, y, z) = T(u, v, w)$. Let S be a set in the uvw-space and assume that the transformation is one-to-one on S except on its boundary. By writing $x = g(u,v,w)$, $y = h(u,v,w)$ and $z = k(u,v,w)$, if f is continuous on $T(S)$, then

$$\iiint_{T(S)} f(x,y,z) dV = \iiint_S f(g,h,k) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV, \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ k_u & k_v & k_w \end{vmatrix}.$$

Particularly, for the spherical coordinate, one has $|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}| = \rho^2 \sin \phi$.

Example 15.30. For the transformation $x = uv$, $y = vw$ and $z = uw$, $|\frac{\partial(x,y,z)}{\partial(u,v,w)}| = 2uvw$.

Example 15.31. Let E be the solid enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Through the change of variables $x = au$, $y = bv$ and $z = cw$, E is the image of $S = \{(u,v,w)|u^2 + v^2 + w^2 \le 1\}$. As $|\frac{\partial(x,y,z)}{\partial(u,v,w)}| = abc$, we have

$$V(E) = \iiint_E dV = abc \iiint_S dV = V(S) = \frac{4\pi abc}{3}.$$