LECTURE NOTES IN CALCULUS I

GUAN-YU CHEN

1. Functions and models

1.1. Notations and terminology. (Sec. 1.1-1.3 in the textbook)

In this notes, we will use the following notations,

- $\mathbb{N} = \{1, 2, 3, ...\}$: The natural numbers.
- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$: The integers.
- $\mathbb{Q} = \{p/q | p, q \in \mathbb{Z}, q \neq 0\}$: The rational numbers.
- $\mathbb{R} =$ completion of $\mathbb{Q}$: The real numbers.
- $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}, i = \sqrt{-1}\}$: The complex numbers.

and logic symbols,

$$\begin{cases} x \in D, & x \text{ belongs to } D,\ x \text{ in } D \\ x \notin D, & x \text{ does not belong to } D \\ \forall x, & \text{for all } x, \text{ for any } x \\ \exists x, & \text{there exists } x \\ \exists! x, & \text{there exists a unique } x \\ p \Rightarrow q, & p \text{ implies } q \\ p \Leftrightarrow q, & p \text{ is equivalent to } q, \text{ or } p \text{ holds if and only if } q \text{ holds} \\ a := b, & a \text{ is defined to be } b \end{cases}$$

For $a, b \in \mathbb{R}$, we write

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}, \quad (a, b] = \{x \in \mathbb{R} | a < x \leq b\}$$

and

$$[a, \infty) = \{x \in \mathbb{R} | x \geq a\}, \quad (-\infty, a] = \{x \in \mathbb{R} | x \leq a\}.$$

The intervals, $[a, b)$, (a, b), (a, ∞) and $(-\infty, a)$, are defined in a similar way.

Definition 1.1. Let D, E be sets. A function $f : D \to E$ is a rule that assigns each element x in D exactly one element, named $f(x)$, in E. Here, $f(x)$ is called the value of f at x, D is called the domain of f and the set $\{f(x) | x \in D\}$ is called the range of f.

In the following, we recall some basic functions.

(1) *Absolute values.* The absolute value of a number a is denoted by $|a|$ and defined by

$$\begin{cases} |a| = a & \text{if } a > 0, \\ |a| = -a & \text{if } a \leq 0. \end{cases}$$

One may regard $|\cdot|$ as a function with domain $\mathbb{R}$ and range $[0, \infty)$.

(2) *Polynomials.* A polynomial is a function P of the following form

$$P(x) = a_0 + \sum_{k=1}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad \forall x \in \mathbb{R},$$

where n is a nonnegative integer, which is called the degree of P when $a_n \neq 0$, and $a_0, ..., a_n$ are real numbers, which are called the coefficients of P. In the case that $n = 2$ and $a_2 \neq 0$, P is also called a quadratic function.

(3) *Power functions.* A power function is a function of the form $f(x) = x^a$, where a is a constant. If $a \in \mathbb{N}$, then $f(x)$ is a polynomial with domain $\mathbb{R}$. If $a = 1/n$ with $n \in \mathbb{N}$, then f is a root function with domain $[0, \infty)$. If $a = -1$, then f is the reciprocal function with domain $(0, \infty)$.

(4) *Rational functions.* A rational function is a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are both polynomials. One can see that the domain of f is $\{x \in \mathbb{R} | q(x) \neq 0\}$.

(5) *Algebraic functions.* An algebraic function is a function constructed by using algebraic operators including addition, subtraction, multiplication, division and taking roots. For example, $\sqrt{x^2+1} - x - 2$ and $(x^{1/3} - 1)/\sqrt{x+4}$ are algebraic functions.

(6) *Trigonometric functions.* Trigonometric functions consist of $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$.

(7) *Exponential functions.* An exponential function is a function of the form $f(x) = b^x$, where $b > 0$ is a constant and called the base. The domain of f is $\mathbb{R}$ and its definition will be discussed in the next subsection.

(8) *Logarithmic functions.* A logarithmic function is a function of the form $f(x) = \log_b x$, where $b > 0$ is called the base. Logarithmic functions are defined to be the inverse function of exponential functions.

Definition 1.2. A function f defined on an interval I is increasing if $x < y$ implies $f(x) < f(y)$ and decreasing if $x < y$ implies $f(x) > f(y)$.

Example 1.1. $f(x) = x^3$ is increasing on $\mathbb{R}$ and $g(x) = \cos x$ is increasing on $[(2k-1)\pi, 2k\pi]$ and decreasing on $[2k\pi, (2k+1)\pi]$ for $k \in \mathbb{Z}$.

Definition 1.3. Let D be a region satisfying $\{-x | x \in D\} = D$. A function f with domain D is even (resp. odd) if $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$) for all $x \in D$.

Example 1.2. $\sin x$ is odd and $\cos x$ is even.

Definition 1.4 (Combination of functions)**.** Let f and g be functions with domains D_f, D_g.

(1) The addition $f + g$ and multiplication fg are defined by

$$(f+g)(x) = f(x) + g(x),\ (fg)(x) = f(x)g(x) \quad \forall x \in D_f \cap D_g.$$

(2) The ratio f/g is a function with domain $\{x \in D_f \cap D_g | g(x) \neq 0\}$ and defined by $(f/g)(x) = f(x)/g(x)$.

(3) The composition of f and g is denoted by $g \circ f$ and defined by

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in D_f,\ f(x) \in D_g.$$

Example 1.3. Let $f(x) = \cos x$ and $g(x) = 2x$. Then, $(f \circ g)(x) = f(g(x)) = f(2x) = \cos(2x)$. To avoid any confusion, we write $\cos(2x)$ for $f \circ g$ instead of $\cos 2x$.