1.2. **Exponential functions.** (Sec. 1.4 in the textbook) Generally, an exponential function is a function of the form

$$f(x) = b^x$$

where b is a positive constant, called the base, and x is any real number.

- In the case that x is an integer, say n, $b^0 = 1$ if $n = 0$ and
$$b^n = \underbrace{b \times b \times \cdots \times b}_{n}, \quad b^{-n} = 1/b^n, \quad \forall n \in \mathbb{N}.$$
- In the case that x is a rational number, say p/q with $p, q \in \mathbb{N}$ and $q \neq 0$,
$$b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p.$$

To see the meaning of b^x when x is irrational, we consider the case $x = \pi$. Note that $\pi = 3.14159\cdots$. By the law of rational exponents, one can show that

$$2^3 < 2^{3.1} < 2^{3.14} < 2^{3.141} < 2^{3.1415} < \cdots$$

and

$$2^4 > 2^{3.2} > 2^{3.15} > 2^{3.142} > 2^{3.1416} > \cdots.$$

Observe that $2^{3.2} - 2^{3.1} = 2^{3.1}(2^{0.1} - 1) < 16(2^{0.1} - 1)$. Similarly, one can show that

$$\begin{aligned} 2^{3.15} - 2^{3.14} &< 16(2^{0.01} - 1) \\ 2^{3.142} - 2^{3.141} &< 16(2^{0.001} - 1) \\ 2^{3.1416} - 2^{3.1415} &< 16(2^{0.0001} - 1) \\ &\vdots \end{aligned}$$

Inductively, we may approximate 2^π using the sequence $2^{3.1}$, $2^{3.14}$, $2^{3.141}$, $2^{3.1415}$, ... or the sequence $2^{3.2}$, $2^{3.15}$, $2^{3.142}$, $2^{3.1416}$, ...

Law of exponents For $a, b \in (0, \infty)$ and $x, y \in \mathbb{R}$, one has

$$a^{x+y} = a^x a^y,\ a^{x-y} = a^x/a^y,\ (a^x)^y = (a^y)^x = a^{xy},\ (ab)^x = a^x b^x.$$

Remark 1.1. Let $f(x) = b^x$. Then, f is increasing if $b > 1$ and decreasing if $0 < b < 1$.

The number e Consider the function $f(x) = b^x$ and let m be the slope of the line "tangent" to the curve $y = f(x)$ at $(0, 1)$. Then, $m > 0$ if $b > 1$; $m < 0$ if $0 < b < 1$; $m = 0$ if $b = 1$. The slope m changes along with the base b and, in fact, the slope m, as a function of b, is increasing. The number e can be defined in any of the following two way.

- e is the base b such that the slope of the line tangent to the curve $y = b^x$ at $(0, 1)$ is 1.
- e is the base b such that the slope of the line tangent to the curve $y = b^x$ at (x, b^x) is exactly b^x for all $x \in \mathbb{R}$.

Note that $e = 2.71828\cdots$.