1.3. **Inverse functions and logarithms.** (Sec. 1.5 in the textbook) Consider the following function. Let $D = \{1, 2, 3\}$, $E = \{a, b, c\}$ and let $f : D \to E$ be a function defined by

$$f(1) = a, \quad f(2) = f(3) = b.$$

Given a, there is only one x such that $f(x) = a$. But, given b, the equation $f(x) = b$ has two solutions, which are 2 and 3.

Definition 1.5. A function f with domain D is one-to-one (briefly, 1-1) or injective if

$$\forall x, y \in D,\, x \neq y \quad \Rightarrow \quad f(x) \neq f(y).$$

Equivalently, $f(x) = f(y)$ implies $x = y$.

Remark 1.2. A function is one-to-one if and only if no horizontal line intersects its graph more than once. This criterion is called the horizontal line test. Equivalently, f is one-to-one if $f(x) = c$ has at most one solution for all $c \in \mathbb{R}$.

Example 1.4. Let $f(x) = x^3$ and $g(x) = x^2$. Clearly, g is not 1-1 because $g(1) = g(-1) = 1$. For f, consider the follow computation,

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) = (x - y)[(x + y)^2 + x^2 + y^2]/2.$$

If $x \neq y$, then $x^2 + y^2 > 0$, which implies $x^3 \neq y^3$. This proves that f is 1-1.

Definition 1.6. Let f be a one-to-one function with domain D and range R. The inverse function of f (denoted by f^{-1} and read as "f inverse") is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y,$$

for all $y \in R$.

Remark 1.3. We write the reciprocal of $f(x)$ as $[f(x)]^{-1}$ or $(1/f)(x)$.

Remark 1.4. If f is 1-1, then the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f. Furthermore, one has $(f^{-1})^{-1} = f$.

Cancellation equations If f is one-to-one with domain D and range R, then

$$(f^{-1} \circ f)(x) = x, \quad \forall x \in D, \quad \text{and} \quad (f \circ f^{-1})(y) = y, \quad \forall y \in R.$$

Example 1.5. Let $f(x) = x + 2$ and $g(x) = x^{1/3}$. Then, $f^{-1}(y) = y - 2$ and $g^{-1}(y) = y^3$.

Computing the inverse function Write $y = f(x)$ and solve x in term of y, say $x = g(y)$. Then, g is the inverse of f.

Example 1.6. Let $f(x) = (x^3 - 2)^{1/5}$ for $x \in \mathbb{R}$. Write $y = f(x)$ and solve this equation to obtain $x = (y^5 + 2)^{1/3}$ for $y \in \mathbb{R}$. Then, the function $f^{-1}(y) = (y^5 + 2)^{1/3}$ is the inverse of f.

Example 1.7. Let $f(x) = \sqrt{x} + 1$ for $x \in [0, \infty)$. Solving $y = f(x)$ yields $x = (y - 1)^2$. This implies $f^{-1}(y) = (y - 1)^2$ with $y \in [1, \infty)$ is the inverse of f.

Remark 1.5. The graph of f^{-1} is the reflection of the graph of f with respect to $y = x$.

Logarithmic functions Let $b > 0$ and $b \neq 1$. Note that $b^x = b^y$ implies $b^{x-y} = 1$ and, hence, $x = y$. This proves that $f(x) = b^x$ is 1-1 with domain $(-\infty, \infty)$ and range $(0, \infty)$. The inverse function of f is called the logarithmic function with base b and denoted by $f^{-1}(y) = \log_b y$ with domain $(0, \infty)$ and range $(-\infty, \infty)$. By definition, one has $\log_b y = x$ if and only if $b^x = y$ for $y \in (0, \infty)$. This implies

$$b^{\log_b y} = y, \quad \forall y \in (0, \infty), \quad \text{and} \quad \log_b(b^x) = x, \quad \forall x \in \mathbb{R}.$$

Law of logarithms For $b, x, y \in (0,\infty)$ and $r \in \mathbb{R}$, one has

$$\log_b(xy) = \log_b x + \log_b y, \quad \log_b(x/y) = \log_b x - \log_b y, \quad \log_b(x^r) = r\log_b x.$$

Remark 1.6. When $b = e$, we also write $\ln x$ for $\log_e x$ and call it the natural logarithm.

Change of base formula Let a, b, x be positive constants and assume that $a, b \neq 1$. Then,

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

In particular, $\log_b x = \ln x / \ln b$.

Proof. It suffices to show the specific case, while the general case follows immediately. Set $y = \log_b x$. Then, $x = b^y = (e^{\ln b})^y = e^{(\ln b)y}$. This implies $(\ln b)y = \ln x$. □

Inverse trigonometric functions It is clear that none of the trigonometric functions is one-to-one on $\mathbb{R}$, unless it is restricted on a specific region. For example, $\sin x$ is one-to-one on $[-\pi/2, \pi/2]$ and we write $\sin^{-1}$ or arcsin for the inverse function, which has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$. This means that, for $x \in [-1, 1]$ and $y \in [-\pi/2, \pi/2]$,

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x.$$

Similarly, the inverse functions of the other trigonometric functions can be defined by

$$\begin{aligned}
\cos^{-1} x = y \quad &\Leftrightarrow \quad \cos y = x, \quad \forall x \in [-1,1], y \in [0,\pi],\\
\tan^{-1} x = y \quad &\Leftrightarrow \quad \tan y = x, \quad \forall x \in \mathbb{R}, y \in (-\pi/2, \pi/2),\\
\sec^{-1} x = y \quad &\Leftrightarrow \quad \sec y = x, \quad \forall y \in [0,\pi/2) \cup (\pi/2, \pi], x \in (-\infty,-1] \cup [1,\infty),\\
\cot^{-1} x = y \quad &\Leftrightarrow \quad \cot y = x, \quad \forall y \in (0,\pi), x \in \mathbb{R},\\
\csc^{-1} x = y \quad &\Leftrightarrow \quad \csc y = x, \quad \forall y \in [\pi/2, \pi) \cup (\pi, 3\pi/2], |x| \geq 1.
\end{aligned}$$

Remark 1.7. Note that the range of $\sec^{-1}$ and $\csc^{-1}$ are not universally agreed on.

Example 1.8. To compute $(\cos \circ \sin^{-1})(\frac{1}{\sqrt{2}})$, note that $\sin^{-1}(\frac{1}{\sqrt{2}}) = \pi/4$. This implies

$$(\cos \circ \sin^{-1})(\tfrac{1}{\sqrt{2}}) = \cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}.$$

Example 1.9. To simplify $\sin(\tan^{-1} x)$, we set $y = \tan^{-1} x$ or, equivalently, $\tan y = x$ with $y \in (-\pi/2, \pi/2)$. Suppose $x > 0$ and consider the right triangle with sides of lengths $1, x$ and $\sqrt{x^2+1}$. Observe that y is exactly the radian of the angle opposite to the side of length x. This gives $\sin(\tan^{-1} x) = \sin y = x/\sqrt{x^2+1}$. For $x < 0$, the discussion is similar and skipped.