10. Parametric equations and polar coordinates

10.1. **Curves defined by parametric equations.** (Sec. 10.1 in textbook)

Definition 10.1. A *parametric curve* on $\mathbb{R}^2$ refers to a curve parametrized by a third variable other than x and y. That is, there are functions f, g with common domain D such that

$$x = f(t), \quad y = g(t), \quad \forall t \in D.$$

These two equations are called *parametric equations* and the parametric curve refers to the set

$$\{(f(t), g(t)) | t \in D\}.$$

The variable t is called the *parameter*.

Remark 10.1. In the case that $D = [a, b]$, $(f(a), g(a))$ is called the *initial point* and $(f(b), g(b))$ is called the *terminal point* of the parametric curve.

Example 10.1. To sketch the parametric curve given by

$$x = t^2 - 2t, \quad y = t + 1,$$

one may write

$$t^2 - 2t = (t + 1)^2 - 4(t + 1) + 3,$$

to get

$$x = y^2 - 4y + 3.$$

The graph of the above equation displays the parametric curve.

Example 10.2. For the parametric equations,

$$x = \cos t, \quad y = \sin t, \quad \forall t \in [0, 2\pi],$$

it is easy to see that its parametric curve is a unit circle, since

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

Note that the initial point is also the end point.

Example 10.3. For the parametric equations,

$$x = \cos t, \quad y = \sin(2t) \quad t \in [0, 2\pi],$$

one may relate x and y through the following equations,

$$y = \begin{cases} 2x\sqrt{1 - x^2} & \forall t \in [0, \pi], \\ -2x\sqrt{1 - x^2} & \forall t \in [\pi, 2\pi]. \end{cases}$$

Example 10.4. Consider a circle of radius r rolls along a straight line. A *cycloid* is the trajectory of a fixed point, say P, on the circumference of a circle. In detail, assume that the circle contact the x-axis at the origin and let $P = (x, y)$ be the point of the circle at the origin. As the circle rotates with angle θ, we obtain

$$x = r\theta - r\sin\theta = r(\theta - \sin\theta), \quad y = r - r\cos\theta = r(1 - \cos\theta).$$