10.2. **Calculus with parametric curves.** (Sec. 10.2 in textbook)

Tangent lines Consider the parametric equation $x = f(t)$ and $y = g(t)$. Assume that the curve can be expressed as $y = F(x)$ and further F, f, g are differentiable. Then, $g(t) = F(f(t))$ and, by the chain rule,

$$g'(t) = F'(f(t))f'(t) \quad \Rightarrow \quad F'(x) = F'(f(t)) = \frac{g'(t)}{f'(t)} \quad \text{if } f'(t) \neq 0.$$

Following the notation of Leibniz, this identity can be rewritten as $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt}$ if $\frac{dx}{dt} \neq 0$. If F, f, g are twice differentiable, then

$$F''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{g''(t)f'(t) - g'(t)f''(t)}{[f'(t)]^3}.$$

Note that $\frac{d^2y}{dx^2} \neq \frac{g''(t)}{f''(t)}$.

Example 10.5. Let $y = F(x)$ be the parametric curve of the cycloid $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$ with $\theta \in \mathbb{R}$. Then, $F'(x) = \frac{\sin\theta}{1-\cos\theta}$ and the slope of the tangent line at $(\pi r, 2r)$ is $F'(\pi r) = 0$.

Area Recall that if F is continuous on $[a, b]$, then the area A bound by $y = F(x)$, $x = a$, $x = b$ and the x-axis equals

$$A = \int_a^b |F(x)|dx = \int_a^b |y|dx.$$

Suppose $y = F(x)$ is given by the parametric curve $x = f(t)$ and $y = g(t)$ and f is increasing with domain $[\alpha, \beta]$. Then,

$$A = \int_{f(\alpha)}^{f(\beta)} F(x)dx = \int_\alpha^\beta F(f(t))f'(t)dt = \int_\alpha^\beta g(t)f'(t)dt.$$

Example 10.6. The area A bounded by the cycloid, $x = 0$, $x = 2\pi r$ and the x-axis is

$$A = r^2\int_0^{2\pi} (1 - \cos\theta)^2 d\theta = r^2\int_0^{2\pi}\left[1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right]d\theta = 3\pi r^2.$$

Arc length Let L be the length of the curve $\{(f(t), g(t))|t \in [\alpha, \beta]\}$. If the curve traverses exactly once and f, g are differentiable on $[\alpha, \beta]$, then

$$L = \lim_{n\to\infty}\sum_{i=1}^n \sqrt{[f(t_{i-1}) - f(t_i)]^2 + [g(t_{i-1}) - g(t_i)]^2},$$

where $t_i = \alpha + i\Delta t$ and $\Delta t = (\beta - \alpha)/n$. By the mean value theorem, we may choose $t_i^*, t_i^{**} \in [t_{i-1}, t_i]$ such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)\Delta t, \quad g(t_i) - g(t_{i-1}) = g'(t_i^{**})\Delta t.$$

If f' and g' are continuous on $[\alpha, \beta]$, then

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}dt.$$

Following the notation of Leibniz, one has

$$L = \int\sqrt{(dx)^2 + (dy)^2} = \int\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

Example 10.7. The length L of the cycloid with $\theta \in [0, 2\pi]$ is equal to

$$L = \int_0^{2\pi} r\sqrt{(1-\cos\theta)^2 + \sin^2\theta}d\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}d\theta.$$

Note that $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$ and $\sin\frac{\theta}{2} \geq 0$ for $\theta \in [0, 2\pi]$. This implies

$$L = 2r\int_0^{2\pi} \sin\frac{\theta}{2}d\theta = 8r.$$

Surface area Let A_x and A_y be the area of the surface obtained by rotating the curve $(f(t), g(t))$ with $t \in [\alpha, \beta]$ about the x-axis and the y-axis. If f, g are continuously differentiable with $g \geq 0$ on $[\alpha, \beta]$ and the curve traverses exactly once, then

$$A_x = \int 2\pi y ds = \int 2\pi y\sqrt{(dx)^2 + (dy)^2} = \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}dt.$$

Similarly, f, g are continuously differentiable with $f \geq 0$ on $[\alpha, \beta]$ and the curve traverses exactly once, then

$$A_y = \int 2\pi x ds = \int 2\pi x\sqrt{(dx)^2 + (dy)^2} = \int_\alpha^\beta 2\pi f(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}dt.$$

Example 10.8. Consider the curve $y = b\sqrt{1 - \frac{x^2}{a^2}}$ with $0 < a < b$ and let A be the area of the surface obtained by rotating the curve $y = b\sqrt{1 - \frac{x^2}{a^2}}$ with $x \in [-a, a]$ about the x-axis. Note that the curve can be parametrized as $x = a\cos\theta$ and $y = b\sin\theta$ with $\theta \in [0, \pi]$. This implies

$$A = \int_0^\pi 2\pi(b\sin\theta)\sqrt{(-a\sin\theta)^2 + (b\cos\theta)^2}d\theta = 2\pi b\int_0^\pi \sin\theta\sqrt{a^2 + (b^2 - a^2)\cos^2\theta}d\theta.$$

By setting $u = \frac{\sqrt{b^2-a^2}}{a}\cos\theta$, one has $du = -\frac{\sqrt{b^2-a^2}}{a}\sin\theta d\theta$ and

$$A = \frac{2\pi a^2 b}{\sqrt{b^2 - a^2}}\int_{-\sqrt{b^2-a^2}/a}^{\sqrt{b^2-a^2}/a} \sqrt{1+u^2}du = \frac{4\pi a^2 b}{\sqrt{b^2 - a^2}}\int_0^{\sqrt{b^2-a^2}/a} \sqrt{1+u^2}du.$$

Recall that

$$\int \sqrt{1+u^2}du = \frac{1}{2}\left(u\sqrt{1+u^2} + \ln|u + \sqrt{1+u^2}|\right) + C.$$

Setting $c = b/a$, leads to

$$A = 2\pi b^2 + 2\pi ab \times \frac{\ln(c + \sqrt{c^2-1})}{\sqrt{c^2-1}}.$$