10.3. **Polar coordinates.** (Sec. 10.3 in textbook)

The *polar* coordinate system is given by, first, choosing a point in the plane as the *pole*, which is usually labelled as O, and then drawing a ray from the pole, which is named the *polar axis*. For any point P on the plane, we use r and θ to denote the length and the radian from the polar axis of $\overrightarrow{OP}$ and call the ordered pair (r, θ) the *polar coordinate* of P. In convention, the angle is positive if it is measured counterclockwise and negative otherwise. If (x, y) is the Cartesian coordinate of P, then

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \Rightarrow \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}. \tag{10.1}$$

It is natural to extend the polar coordinate (r, θ) to the case that r is negative by identifying (r, θ) with $(-r, \theta + \pi)$ when $r < 0$. Clearly, (10.1) is preserved under such a generalization.

The graph of a polar equation, $r = f(\theta)$ or $\theta = g(r)$ or more generally $F(r, \theta) = 0$, is called a *polar curve*.

Example 10.9. Consider the polar curves of (1) $r = 2$, (2) $\theta = \pi/6$, (3) $r = \sin\theta$, (4) $r = 1+\sin\theta$ and (5) $r = \cos 2\theta$. Let (x, y) be the Cartesian coordinate of these curves. For (1), it is clear that $x^2+y^2 = 4$. For (2), $x = r\cos(\pi/6) = \frac{\sqrt{3}r}{2}$ and $y = r\sin(\pi/6) = \frac{r}{2}$. This implies $x = \sqrt{3}y$ with $y \in \mathbb{R}$. For (3), one has $x^2 + y^2 = r^2 = r\sin\theta = y$ or $x^2 + (y - 1/2)^2 = 1/4$, which is a circle of radius $1/2$ and centered at $(0, 1/2)$. For (4) and (5), we refer the read to the textbook for details. It's worthwhile to know that (4) is called *cardioid* and (5) is a *four-leaved rose*.

Symmetry of polar curves

(1) If θ is replaced by $-\theta$, then the curve is symmetric about the polar axis.
(2) If r is replaced by $-r$ or when θ is replaced by $\theta + \pi$ or $\theta - \pi$, then the curve is symmetric about the pole.
(3) If θ is replaced by $\pi - \theta$, then the curve is symmetric about the line $\theta = \pi/2$.

Tangents to polar curves For the polar curve $r = f(\theta)$, if f is differentiable, then the tangent line to $r = f(\theta)$ at the point $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$ is equal to

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 10.10. The tangent line to the curve $r = 1/\theta$ at $\theta = \pi$ has slope

$$\frac{dy}{dx} = \frac{\frac{dr\sin\theta}{d\theta}}{\frac{dr\cos\theta}{d\theta}} = \frac{\theta\cos\theta - \sin\theta}{-\theta\sin\theta - \cos\theta}.$$

For $\theta = \pi$, the tangent line to the curve at $(-1/\pi, 0)$ is $y = (-\pi)(x + 1/\pi) = -\pi x - 1$.

Example 10.11. For the cardioid $r = 1 + \sin\theta$, the slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{\frac{d(\sin\theta+\sin^2\theta)}{d\theta}}{\frac{d(\cos\theta+\frac{1}{2}\sin 2\theta)}{d\theta}} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}.$$

When $\theta = \pi/3$, $dy/dx = -1$ and the tangent line is $x + y = (5 + 3\sqrt{3})/4$.