10.4. **Areas and lengths in polar coordinates.** (Sec. 10.4 in textbook)

Area Let A be the area of the region enclosed by the polar curves $r = f(\theta)$, $\theta = a$ and $\theta = b$ with $a < b$. To compute A, we follow the idea of Riemann sum approximation. Set $\Delta\theta = (b-a)/n$, $\theta_i = a + i\Delta\theta$ and select $\theta_i^* \in [\theta_{i-1}, \theta_i]$. Note that the area of the region enclosed by $r = f(\theta)$, $\theta = \theta_{i-1}$ and $\theta = \theta_i$ is roughly the area of the sector of a circle with radius $|f(\theta_i^*)|$ and angle $\Delta\theta$, which is equal to $\frac{1}{2}|f(\theta)|^2\Delta\theta$. As a result, if f is continuous on $[a, b]$, then

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}f(\theta_i^*)^2\Delta\theta = \int_a^b \frac{1}{2}f(\theta)^2 d\theta = \int_a^b \frac{1}{2}r^2 d\theta.$$

Example 10.12. Consider the area enclosed by the four-leaved rose $r = \cos 2\theta$. By the symmetry of the curve, the area is four times of the area bounded by $r = \cos 2\theta$ with $\theta \in [-\pi/4, \pi/4]$. This implies

$$A = 4\int_{-\pi/4}^{\pi/4}\frac{1}{2}\cos^2 2\theta d\theta = \int_{-\pi/4}^{\pi/4}(1 + \cos 4\theta)d\theta = \frac{\pi}{2}.$$

Example 10.13. Consider the area inside the circle $r = 3\sin\theta$ but outside the cardioid $r = 1 + \sin\theta$. In some computations, one has $3\sin\theta = 1 + \sin\theta$ if and only if $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. This implies $3\sin\theta \geq 1 + \sin\theta$ if and only if $\theta \in [\pi/6, 5\pi/6]$. As a consequence, the desired area equals

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6}[(3\sin\theta)^2 - (1+\sin\theta)^2]d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}[8\sin^2\theta - 2\sin\theta - 1]d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}[3 - 4\cos 2\theta - 2\sin\theta]d\theta = \frac{1}{2}(3\theta - 2\sin 2\theta + 2\cos\theta)\Big|_{\pi/6}^{5\pi/6} = \pi.$$

Arc length Recall that the length of a parametric curve is $L = \int ds = \int\sqrt{(dx)^2 + (dy)^2}$. For the polar curve $r = f(\theta)$, one has $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. This implies

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2,$$

and then the length of the curve $r = f(\theta)$ with $\theta \in [a, b]$ equals

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta.$$

Example 10.14. Consider the cardioid $r = 1 + \sin\theta$. The curve has length

$$\int_0^{2\pi}\sqrt{(1+\sin\theta)^2 + \cos^2\theta}d\theta = \int_0^{2\pi}\sqrt{2 + 2\sin\theta}d\theta.$$

By writing $\sqrt{2 + 2\sin\theta} = \frac{2|\cos\theta|}{\sqrt{2 - 2\sin\theta}}$, we have

$$\int_0^{2\pi}\sqrt{2 + 2\sin\theta}d\theta$$

$$= \int_0^{\pi/2}\frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}}d\theta + \int_{3\pi/2}^{2\pi}\frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}}d\theta - \int_{\pi/2}^{3\pi/2}\frac{2\cos\theta}{\sqrt{2 - 2\sin\theta}}d\theta$$

$$= -2\sqrt{2 - 2\sin\theta}\Big|_0^{\pi/2} - 2\sqrt{2 - 2\sin\theta}\Big|_{3\pi/2}^{2\pi} + 2\sqrt{2 - 2\sin\theta}\Big|_{\pi/2}^{3\pi/2} = 8.$$