2. Limits and derivatives

2.1. The limit of a function. (Sec. 2.2 in the textbook) Let $f(x) = x^2 - x + 2$ for $x \neq 2$ and consider the value of f when x is close to 2.

x	$f(x)$	x	$f(x)$
1.9	3.71	2.1	4.31
1.99	3.9701	2.01	4.0301
1.999	3.997001	2.001	4.003001

From the table, one can see that the value of $f(x)$ approaches 4 when x gets close to 2.

Definition 2.1 (Informal definition of limits)**.** Let f be a function defined on a neighborhood of a except at a. We write

$$\lim_{x\to a} f(x) = L, \quad \text{or} \quad f(x) \to L \quad \text{as} \quad x \to a,$$

and say

the limit of $f(x)$ equals L as x approaches a,

if the values of $f(x)$ can be arbitrarily close to L as x is sufficiently close to a, but not equal to a. Here, L is called the limit of f at a.

Remark 2.1. Note that the value and limit of f at a are unrelated.

Example 2.1. Consider the following two functions.

$$f(x) = \frac{x-1}{x^2-1}, \quad g(x) = \frac{1}{x+1}.$$

Note that the domains of f and g are $\mathbb{R} \setminus \{\pm 1\}$ and $\mathbb{R} \setminus \{-1\}$ respectively. Since f and g are equal on a neighborhood of 1 except at 1, their limits at 1 coincide.

Example 2.2. Let $f(x) = \sin(1/x)$ at 0. Note that

$$f(1/n\pi) = 0, \quad f(1/(2n\pi + \pi/2)) = 1, \quad \forall n \in \mathbb{N}.$$

This means that the value of $f(x)$ can be arbitrarily close to 0 and 1 as x follows specific sequences tending to 0. Hence, the limit of f at 0 does not exist. We will show later that the limit of f exists at any point other than 0.

Example 2.3. Consider the following function.

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

The value of f is close to 0 and 1 when x approaches 0 from the left and right sides. This implies the limit of f at 0 does not exist.

Definition 2.2 (Informal definition of one-sided limits)**.** We write

$$\lim_{x\to a^-} f(x) = L,$$

and say

the left-hand limit of $f(x)$ as x approaches a,

or

the limit of $f(x)$ as x approaches a from the left,

is equal to L if the values of $f(x)$ can be arbitrarily close to L when x is sufficiently close to a and less than a.

The informal definition of the right limit is obtained by replacing a^-, "left" and "less" with a^+, "right" and "greater". An immediate result of the above definition says

$$\lim_{x\to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Example 2.4. For the function $f(x) = 1/|x|$, it is clear that the value of $f(x)$ can be positive and arbitrarily large, as x is close to 0. By the definition of limits, f has no limit at 0. In this case, we write

$$\lim_{x\to 0} f(x) = \infty,$$

where ∞ is called the infinity and refers to the concept of being arbitrarily large, becoming infinite or increasing without bound.

Definition 2.3 (Informal definition of tending to infinity)**.** For a function f defined on both side of a, we write

$$\lim_{x\to a} f(x) = \infty, \quad \text{or} \quad f(x) \to \infty \quad \text{as} \quad x \to a,$$

if the values of $f(x)$ can be arbitrarily large as x is sufficiently close to a but not equal to a.

Remark 2.2. We use $-\infty$ to denote the concept of being arbitrarily large negative or decreasing without bound, and write

$$\lim_{x\to a} f(x) = -\infty, \quad \text{when} \quad \lim_{x\to a}(-f(x)) = \infty.$$

Remark 2.3. For the following one-sided limits, one may define them in a similar way as before.

$$\lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty.$$

Example 2.5. Let $f(x) = \frac{x}{x^3-1}$. Note that $x^3 - 1 \to 0$ as $x \to 1$. This implies that $f(x)$ can be arbitrarily large when $x \to 1^+$, and can be arbitrarily large negative when $x \to 1^-$. Hence, we may conclude that

$$\lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to 1^-} f(x) = -\infty.$$

Definition 2.4. A line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following conditions holds.

$$\lim_{x\to a^+} f(x) = \infty, \ \lim_{x\to a^-} f(x) = \infty, \ \lim_{x\to a^+} f(x) = -\infty, \ \lim_{x\to a^-} f(x) = -\infty.$$

Example 2.6. Let $f(x) = \sec(x^2)$. Note that $\cos(x^2) = 0$ if and only if $x^2 = (n - 1/2)\pi$ for $n \in \mathbb{N}$. This implies $x = \pm\sqrt{(n - 1/2)\pi}$ are vertical asymptotes of $y = f(x)$ for $n \in \mathbb{N}$.