2.2. **Calculating limits using the limit laws.** (Sec. 2.3 in the textbook)

Limit laws Let c be a constant and assume the existence of the limits of f, g at a. Then,

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x), \quad \lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x),$$

and

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x), \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0.$$

Remark 2.4. Note that the limit laws also apply for the one-sided limits.

Based on the following two fundamental limits,

$$\lim_{x\to a} c = c, \quad \lim_{x\to a} x = a,$$

one may derive that, for any polynomial $P(x)$,

$$\lim_{x\to a} P(x) = P(a).$$

Proposition 2.1 (Direct substitution property)**.** *Let f be a rational function with domain D. For $a \in D$, $f(x) \to f(a)$ as $x \to a$.*

Power law and root law Let $n \in \mathbb{N}$ and assume that $\lim_{x\to a} f(x)$ exists. Then,

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n, \quad \lim_{x\to a}[f(x)]^{1/n} = \left[\lim_{x\to a} f(x)\right]^{1/n},$$

where the latter equality requires $f(x) \geq 0$ for x is close enough to a, when n is even.

Remark 2.5. The direct substitution property also applies for algebraic functions on their domains.

Example 2.7. Let $f(x) = (x^2 - 2x + 3)^{1/2}$. Since $x^2 - 2x + 3 \to 2 > 0$ as $x \to 1$, the limit of f at 1 is $\sqrt{2} = f(1)$. In fact, f is an algebraic function with domain $\mathbb{R}$ and the desired limit can be given by Remark 2.5.

Example 2.8. To compute the limit of $f(x) = \frac{\sqrt{x+1}-1}{x}$ at 0, we write

$$f(x) = \frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{x+1}+1} =: g(x), \quad \forall x \neq 0.$$

Since $f(x) = g(x)$ for $x \neq 0$ and the domain of g is $\mathbb{R}$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = g(0) = 1/2.$$

Theorem 2.2. *Assume $f(x) \leq g(x)$ as x is close to a and the limits of f, g at a exist. Then,*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem 2.3 (The Squeeze theorem)**.** *Assume that $f(x) \leq g(x) \leq h(x)$ as x is close to a and that the limits of f, h at a exist and equal L. Then, $g(x) \to L$ as $x \to a$.*

Example 2.9. To compute the limit of $f(x) = x^2 \sin(1/x)$ at 0, note that $|\sin(1/x)| \leq 1$. This implies

$$-x^2 \leq x^2 \sin(1/x) \leq x^2, \quad \forall x \in \mathbb{R}.$$

Since $x^2 \to 0$ as $x \to 0$, the squeeze theorem implies that $f(x) \to 0$ as $x \to 0$.

Remark 2.6. Theorems 2.2-2.3 also hold for one-sided limits.