2.3. **The precise definition of a limit.** (Sec. 2.4 in textbook)

Let's start with the following function

$$f(x) = \begin{cases} 3x - 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Intuitively, as x approaches 1 but not equal 1, the values of $f(x)$ is getting close to 2. To show that $f(x)$ can be arbitrarily close to 2 as x is sufficiently close to 1 but not equal to 1, we first let 0.1 be the tolerance and our goal is to find $\delta > 0$ such that

$$0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 2| < 0.1. \tag{2.1}$$

Consider the following computation,

$$|f(x) - 2| < 0.1 \quad \forall x \neq 1 \quad \Leftrightarrow \quad 3|x - 1| < 0.1 \quad \forall x \neq 1 \quad \Leftrightarrow \quad 0 < |x - 1| < 0.1/3.$$

Immediately, this implies that any δ less than or equal $0.1/3$ is sufficient for (2.1). In a similar way, if the tolerance is respectively replaced by 0.01 and 0.001, then δ can be selected as any positive real number respectively less than $0.01/3$ and $0.001/3$.

In general, if ϵ is the tolerance, then δ can be selected as any positive real number less than or equal to $\epsilon/3$. For convenience, we set $\delta = \epsilon/3$ for any $\epsilon > 0$. In the setting, one has

$$0 < |x - 1| < \delta \quad \Rightarrow \quad 0 < |x - 1| < \epsilon/3 \quad \Rightarrow \quad |f(x) - 2| < \epsilon. \tag{2.2}$$

Note that, in the above reasoning, $\Rightarrow$ is in fact $\Leftrightarrow$.

Remark 2.7. Note that, in (2.2), δ can be replaced by any positive real number less than $\epsilon/3$, say $\epsilon/4$ or $\epsilon/5$.

Definition 2.5 (Limits)**.** Let f be a function defined on some open interval containing a except probably at a. The limit of $f(x)$ as x approaches a is L if, for any $\epsilon > 0$ (arbitrarily close), there is $\delta > 0$ (sufficiently close) such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{2.3}$$

In this case, we write

$$\lim_{x \to a} f(x) = L.$$

Remark 2.8. Let $a, c \in \mathbb{R}$ and $f(x) = c$ and $g(x) = x$. Note that, for $\epsilon > 0$,

$$|f(x) - c| = 0 < \epsilon, \quad \forall x \in \mathbb{R}, \quad |g(x) - a| = |x - a| < \epsilon, \quad \forall |x - a| < \epsilon.$$

This implies $f(x) \to f(a)$ and $g(x) \to g(a)$ as $x \to a$.

Remark 2.9. Note that if (2.3) holds for (δ, ϵ), then it holds for any pairs (δ', ϵ) and (δ, ϵ') with $\delta' < \delta$ and $\epsilon' > \epsilon$. Hence, to prove (2.3), it loses no generality to assume that ϵ is less then some fixed constant, say $\epsilon < 1$.

Example 2.10. Let $f(x) = \sqrt{x}$ for $x \geq 0$. Obviously, one can guess from the graph $y = f(x)$ that the limit of f at 4 equals 2. We demonstrate two rigorous proofs in the following.

Proof 1: The proof is given during the search of δ. We start with the requirement $|f(x) - 2| < \epsilon$. Note that f is defined on $[0, \infty)$ and $|f(0) - 2| = 2$. For $x \geq 0$ and $\epsilon < 2$, one may compute

$$|\sqrt{x} - 2| < \epsilon \quad \Leftrightarrow \quad 2 - \epsilon < \sqrt{x} < 2 + \epsilon \quad \Leftrightarrow \quad \epsilon(\epsilon - 4) < x - 4 < \epsilon(\epsilon + 4).$$

When $0 < \epsilon < 1$, it is easy to check that $\epsilon(\epsilon - 4) < -3\epsilon$ and $\epsilon(\epsilon + 4) > 3\epsilon$. Immediately, this implies that, for $\epsilon \in (0, 1)$,

$$-3\epsilon < x - 4 < 3\epsilon \quad \Rightarrow \quad \epsilon(\epsilon - 4) < x - 4 < \epsilon(\epsilon + 4) \quad \Leftrightarrow \quad |\sqrt{x} - 2| < \epsilon.$$

As a result, we may conclude from the above computations that, for $\epsilon \in (0,1)$ and $\delta = 3\epsilon$,

$$0 < |x-4| < \delta \quad \Rightarrow \quad |f(x)-2| < \epsilon.$$

Proof 2: Provide a reasoning of (2.3) with prescribed δ. Let $0 < \epsilon < 1$ and $\delta = 3\epsilon$. Observe that if $0 < |x-4| < \delta$, then $1 < x < 7$, which implies

$$|f(x)-2| = |\sqrt{x}-2| = \frac{|(\sqrt{x}-2)(\sqrt{x}+2)|}{\sqrt{x}+2} = \frac{|x-4|}{\sqrt{x}+2} < \frac{\delta}{\sqrt{1}+2} = \epsilon,$$

as desired.

Definition 2.6 (One-sided limits)**.** The left-hand limit of $f(x)$ as x approaches a equals L if, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$a - \delta < x < a \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

In this case, we write

$$\lim_{x \to a^-} f(x) = L.$$

The definition for the right-hand limit is given by replacing $a - \delta < x < a$ and a^- with $a < x < a + \delta$ and a^+.

Example 2.11. Let f be a function defined by

$$f(x) = \begin{cases} 1-2x & \text{if } x < 1, \\ 0 & \text{if } x = 1, \\ 3x-2 & \text{if } x > 1. \end{cases}$$

Note that, for any $\epsilon > 0$,

$$1 - \epsilon/2 < x < 1 \quad \Rightarrow \quad |f(x) - (-1)| < \epsilon,$$

and

$$1 < x < 1 + \epsilon/3 \quad \Rightarrow \quad |f(x) - 1| < \epsilon.$$

This implies the left-hand and right-hand limits of f at 1 are -1 and 1.

Next, we prove some limit laws.

Proof of the limit law for addition. Let L, M be the limits of f, g at a. For $\epsilon > 0$, we may choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L| < \epsilon/2, \quad \forall 0 < |x-a| < \delta_1, \quad |g(x) - M| < \epsilon/2, \quad \forall 0 < |x-a| < \delta_2.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x-a| < \delta$,

$$|(f+g)(x) - (L+M)| \leq |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Proof of the limit law for multiplication. Let L, M be the limits of f, g at a. For $\epsilon > 0$, we may choose $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_1 \quad \Rightarrow \quad |f(x) - L| < \frac{\epsilon}{|L| + |M| + 1}, \tag{2.4}$$

and

$$0 < |x-a| < \delta_2 \quad \Rightarrow \quad |g(x) - M| < \frac{\epsilon}{|L| + |M| + 1}. \tag{2.5}$$

Assume $\epsilon < 1$, set $\delta = \min\{\delta_1, \delta_2\}$ and let $0 < |x-a| < \delta$. Note that

$$|f(x)g(x) - LM| = |(f(x) - L)g(x) + L(g(x) - M)| \leq |f(x) - L||g(x)| + |L||g(x) - M|.$$

By (2.4)-(2.5), we have

$$|f(x) - L| < \epsilon/(|L| + |M| + 1), \quad |g(x) - M| < \epsilon/(|L| + |M| + 1).$$

The second inequality implies $|g(x)| < M + \epsilon/(|L| + |M| + 1)$, which yields

$$|f(x)g(x) - LM| \leq \frac{\epsilon}{|L| + |M| + 1}\left(|M| + \frac{\epsilon}{|L| + |M| + 1}\right) + |L|\frac{\epsilon}{|L| + |M| + 1} < \epsilon.$$

□

Proof of the limit law for division. It suffices to show that if $g(x) \to L \neq 0$ as $x \to a$, then $1/g(x) \to 1/L$ as $x \to a$. For $0 < \epsilon < 1/|L|$, we may select $\delta > 0$ such that $|g(x) - L| < \epsilon L^2/2$ for $0 < |x - a| < \delta$. This implies, when $0 < |x - a| < \delta$,

$$|g(x) - L| < \frac{\epsilon L^2}{2} < \frac{|L|}{2} \quad \Rightarrow \quad |g(x)| > \frac{|L|}{2} > 0,$$

and, then,

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| = \frac{|g(x) - L|}{|g(x)| \times |L|} < \frac{\epsilon L^2/2}{L^2/2} = \epsilon.$$

□

Definition 2.7 (Infinite limits)**.** Let f be a function defined on an open interval containing a, except possibly at a. We write

$$\lim_{x \to a} f(x) = \infty$$

if, for any $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) > M.$$

Similarly, we write

$$\lim_{x \to a} f(x) = -\infty$$

if, for any $N > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad f(x) < -N.$$

Example 2.12. Let $f(x) = 1/\sqrt{|x|}$. To show that f has an infinite limit at 0, let $M > 0$ and set $\delta = M^{-2}$. For $0 < |x| < \delta$, one has $\sqrt{|x|} < M^{-1}$, which implies $f(x) = 1/\sqrt{|x|} > M$. Hence, we obtain $f(x) \to \infty$ as $x \to 0$.

Remark 2.10. For the one-sided infinite limit, we say that $f(x) \to \infty$ as $x \to a^+$ if, for any $M > 0$, there is $\delta > 0$ such that

$$a < x < a + \delta \quad \Rightarrow \quad f(x) > M.$$

The other three infinite limits are defined in a similar way and omitted.