2.4. **Continuity.** (Sec. 2.5 in the textbook)

Definition 2.8. A function is continuous at a if

$$\lim_{x\to a} f(x) = f(a) = f\left(\lim_{x\to a} x\right),$$

or, equivalently, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon, \quad \forall |x - a| < \delta.$$

Remark 2.11. If f is continuous at a, then

(1) f is defined on an open interval containing a.
(2) The limit of $f(x)$ as x approaches a exists and equals to $f(a)$.

Remark 2.12. A function that is not continuous at a is called discontinuous at a.

Remark 2.13. Let f be a function defined on an open interval containing a. Then, f is discontinuous at a if and only if either of the following holds.

(1) The limit of f at a does not exist.
(2) The limit of f at a exists but doesn't equal $f(a)$, including the case that $f(a)$ isn't defined.

Specific discontinuity A function f has a

(1) removable discontinuity at a if
$$\lim_{x\to a} f(x) = L,$$
but either f is not defined at a or $f(a) \neq L$.
(2) jump discontinuity at a if
$$\lim_{x\to a^-} f(x) = L, \quad \lim_{x\to a^+} f(x) = R,$$
but $L \neq R$.
(3) infinite discontinuity at a if either of the following conditions holds,
$$\lim_{x\to a^-} f(x) \in \{\pm\infty\} \quad \text{and} \quad \lim_{x\to a^+} f(x) \in \{\pm\infty\}.$$

Example 2.13. Let $f(x) = (x^2 - 1)/[(x-1)(x-2)]$ with domain $\mathbb{R} \setminus \{1, 2\}$. Since $f(x) = g(x) := (x+1)/(x-2)$ for $x \notin \{1, 2\}$, the limits of f, g at any point other than $1, 2$ agree with each other. Note that $g(x) \to -2$ as $x \to 1$ and $g(x) \to \infty$ as $x \to 2^+$ and $g(x) \to -\infty$ as $x \to 2^-$. This implies that f has a removable discontinuity at 1 and an infinite discontinuity at 2. It's worthwhile to remark that g is continuous at 1.

Remark 2.14. There are examples of which discontinuity is not of any type introduced above. For instance, consider the function $f(x) = \sin(1/x)$ for $x \neq 0$. One can show that the right-hand and left-hand limits of f at 0 neither exist nor become infinity.

Definition 2.9. A function f is continuous from the right or right-continuous at a if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or equivalently} \quad \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(a)| < \epsilon, \forall a \leq x < a + \delta.$$

f is continuous from the left or left-continuous at a if

$$\lim_{x\to a^-} f(x) = f(a) \quad \text{or equivalently} \quad \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(a)| < \epsilon, \forall a - \delta < x \leq a.$$

Remark 2.15. A function is continuous at a if and only if it is continuous from the left and from the right at a.

Example 2.14. Let $f(x) = [x]$, where $[x]$ is the greatest integer among all integers less than or equal to x. For $a \notin \mathbb{Z}$, $f(x) = f(a)$ for $[a] < x < [a] + 1$. This implies that f is continuous at a for $a \notin \mathbb{Z}$. For $a \in \mathbb{Z}$, one has

$$\lim_{x\to a^+} f(x) = a = f(a), \quad \lim_{x\to a^-} f(x) = a - 1 \neq f(a).$$

This implies that f has a jump discontinuity at a. In fact, f is continuous from the right at a but discontinuous from the left at a.

Definition 2.10. A function is continuous on an interval if it is continuous at every point of that interval, while the continuity at boundary points refers to the left or right continuity.

Example 2.15. Consider the function $f(x) = 1 - \sqrt{1 - x^2}$ with domain $[-1, 1]$. By the direct substitution property for algebraic functions, f is continuous on $[-1, 1]$.

Theorem 2.4. *Let c, d be constants and assume that f, g are continuous at a. Then, $cf + dg$, fg and f/g are continuous at a, where f/g requires $g(a) \neq 0$.*

Proof. By the limit laws, we have

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x) = f(a)g(a)$$

and

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) = cf(a), \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}.$$

□

Remark 2.16. Note that Theorem 2.4 also holds for right-continuous and left-continuous.

Applying Theorem 2.4 to the function $x \mapsto x$ yields the following theorem.

Theorem 2.5. *Polynomials and rational functions are continuous on their domains.*

Remark 2.17. By the root law, all root functions are continuous on their domains. Immediately, this implies that algebraic functions are also continuous on their domains.

Example 2.16. Let $f(x) = \frac{x^2+x+1}{3-x}$. Since f is a rational function of which domain contains 2, the limit of f at 2 equals $f(2) = 7$.

For trigonometric functions, one can see from the geometric definition that $0 \leq \sin x \leq x$ for $0 \leq x \leq \pi/2$. By the squeeze theorem, the right-hand limit of $\sin x$ at 0 is 0. Since $\sin x$ is an odd function, the left-hand limit of $\sin x$ at 0 is also 0. As a consequence, this implies

$$\lim_{x\to 0} \sin x = 0 = \sin 0,$$

and, thus, $\sin x$ is continuous at 0. In addition with the identity, $\cos x = \sqrt{1 - \sin^2 x}$ for $x \in [-\pi/2, \pi/2]$, one may conclude that $\cos x$ is continuous at 0. Let $a \in \mathbb{R}$ and recall the following formulas,

$$\sin(x + a) = \sin x \cos a + \cos x \sin a, \quad \cos(x + a) = \cos x \cos a - \sin x \sin a.$$

By the limit laws, we obtain

$$\lim_{x\to a} \sin x = \lim_{y\to 0} \sin(y + a) = \sin 0 \cos a + \cos 0 \sin a = \sin a$$

and

$$\lim_{x\to a} \cos x = \lim_{y\to 0} \cos(y + a) = \cos 0 \cos a - \sin 0 \sin a = \cos a.$$

This proves that $\sin x$ and $\cos x$ are continuous everywhere.

For exponential functions, let $f(x) = 2^x$. Note that $1 + s \le (1 + s/n)^n$ for $s \ge 0$. Clearly, one has $f(1/n) \le 1 + 1/n$. By the law of exponent and the squeeze theorem, the right limit of f at 0 equals 1. In addition with the identity $f(-x) = 1/f(x)$, we may conclude that the left-hand limit of f at 0 also equals 1. This implies that f is continuous at 0 and, then, for $a \in \mathbb{R}$,

$$\lim_{x \to a} f(x) = \lim_{y \to 0} f(y + a) = \lim_{y \to 0} f(a)f(y) = f(a).$$

This proves that f is continuous on $\mathbb{R}$. The general case can be proved in a similar way and the details are omitted.

We summarize the above discussions in the following theorem.

Theorem 2.6. *Trigonometric and exponential functions are continuous on their domains.*

The next two theorems concern the composition and inverse of continuous functions.

Theorem 2.7. *If f is continuous at b and $\lim_{x \to a} g(x) = b$, then*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

In particular, if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Proof. Let $\epsilon > 0$. Since f is continuous at b, one may choose $\delta > 0$ such that $|f(y) - f(b)| < \epsilon$ for $|y - b| < \delta$. For such δ, since g has limit b at a, one may choose $\eta > 0$ such that $|g(x) - b| < \delta$ for $0 < |x - a| < \eta$. This implies that $|f(g(x)) - f(b)| < \epsilon$ for $0 < |x - a| < \eta$. □

Theorem 2.8. *Let f be a function defined on an open interval I and $a \in I$. If f is one-to-one and continuous on I, then f^{-1} is continuous on $f(I)$.*

The proof of Theorem 2.8 is based on the intermediate value theorem (see below) and will be discussed in the end of this subsection.

Corollary 2.9. *Root functions, logarithmic functions and inverse trigonometric functions are continuous on their domains.*

Example 2.17. Let $f(x) = \sin^{-1} x$ and $g(x) = \frac{1-\sqrt{x}}{1-x}$. Note that $g(x) = 1/(1 + \sqrt{x})$ for $x > 0$ and $x \neq 1$. This implies

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}.$$

By Theorem 2.7, since $1/2$ is contained in the domain of f, we have

$$\lim_{x \to 1} f(g(x)) = f(1/2) = \pi/6.$$

Theorem 2.10 (The intermediate value theorem)**.** *Suppose that f is continuous on $[a, b]$ and let C be a number between $f(a)$ and $f(b)$ (that is, $f(a) < C < f(b)$ or $f(b) < C < f(a)$). Then, there exists $c \in (a, b)$ such that $f(c) = C$.*

Corollary 2.11. *If f is a continuous function on $[a, b]$ with $f(a)f(b) < 0$, then there must be a root of the equation $f(x) = 0$ between a and b.*

Proof of Theorem 2.10. Set $\alpha = f(a)$ and $\beta = f(b)$. Without loss of generality, we assume that $\alpha < \beta$. (Otherwise, one may consider the function $g := -f$.) Set $a_1 = a$ and $b_1 = b$. If $f(\frac{a+b}{2}) \ge C$, we set $a_2 = a_1$ and $b_2 = (a + b)/2$. If $f(\frac{a+b}{2}) < C$, we set $a_2 = (a + b)/2$ and $b_2 = b_1$. Inductively, we define

$$a_{n+1} = \begin{cases} a_n & \text{if } f(\frac{a_n+b_n}{2}) \ge C \\ (a_n + b_n)/2 & \text{if } f(\frac{a_n+b_n}{2}) < C \end{cases}, \quad b_{n+1} = \begin{cases} (a_n + b_n)/2 & \text{if } f(\frac{a_n+b_n}{2}) \ge C \\ b_n & \text{if } f(\frac{a_n+b_n}{2}) < C \end{cases}.$$

In the above setting, it is easy to see that $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all $n \ge 1$, $b_n - a_n = (b-a)2^{1-n}$ and $f(a_n) \le C \le f(b_n)$. By the completion of real numbers, there is a constant $c \in [a, b]$ such that $a_n \le c \le b_n$ for all $n \ge 1$ and the sequences, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, converge to c. By the continuity of f, this implies

$$C \ge \lim_{n\to\infty} f(a_n) = f(c), \quad C \le \lim_{n\to\infty} f(b_n) = f(c),$$

which yields $f(c) = C$. □

Proof of Theorem 2.8. Let $C \in f(I)$ and $c \in I$ such that $C = f(c)$. Set $g = f^{-1}$. For $\epsilon > 0$, choose $x_1 \in (c-\epsilon, c)$ and $x_2 \in (c, c+\epsilon)$ and define

$$\alpha := f(x_1), \quad \beta := f(x_2).$$

Without loss of generality, we may assume that $\alpha < \beta$. (Otherwise, one may consider the function $x \mapsto -f(x)$.) Since f is one-to-one and continuous on I, the intermediate value theorem implies that $\alpha < C < \beta$ and $(\alpha, \beta) \subset f((x_1, x_2))$. As a consequence, this leads to

$$g((\alpha, \beta)) \subset g(f((x_1, x_2))) = (x_1, x_2) \subset (c-\epsilon, c+\epsilon).$$

By setting $\delta = \min\{C - \alpha, \beta - C\}$, we may conclude that $|g(y) - c| < \epsilon$ for $|y - C| < \delta$. □

Remark 2.18. By a similar proof as above, one can show that if f is one-to-one and continuous on I, then f is either increasing or decreasing.