2.5. **Limits at infinity: Horizontal asymptotes.** (Sec. 2.6 in the textbook)

In this subsection, we discuss the behavior of a function $f(x)$ as x tends to $\pm\infty$. We start by considering the function $f(x) = \frac{x^2-1}{x^2+1}$. One can see from its graph $y = f(x)$ that $f(x)$ gets close to 1 as x tends to ∞.

Definition 2.11. Let f be a function defined on some interval (a, ∞). We write

$$\lim_{x\to\infty} f(x) = L, \tag{2.6}$$

if the values of $f(x)$ can be arbitrarily close to L by taking x sufficiently large. More precisely, we write (2.6) if, for any $\epsilon > 0$, there is $M > 0$ such that

$$|f(x) - L| < \epsilon, \quad \forall x > M.$$

Remark 2.19. In a similar way, we write $\lim_{x\to-\infty} f(x) = L$ if the values of $f(x)$ can be arbitrarily close to L by taking x sufficiently large negative. Or, equivalently, for any $\epsilon > 0$, there is $N > 0$ such that $|f(x) - L| < \epsilon$ for all $x < -N$.

Example 2.18. Let $f(x) = \frac{x^2-1}{x^2+1}$. To prove that the limit of f at ∞ equals 1, let $\epsilon > 0$ and $M = \sqrt{2/\epsilon}$. Then, for $x > M$,

$$|f(x) - 1| = \frac{2}{x^2+1} < \frac{2}{x^2} < \frac{2}{M^2} = \epsilon.$$

Definition 2.12. A line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if any of the following limits holds,

$$\lim_{x\to\infty} f(x) = L, \quad \lim_{x\to-\infty} f(x) = L.$$

Remark 2.20. There are at most two horizontal asymptotes for each curve $y = f(x)$.

Lemma 2.12. *For any function f, one has*

$$\lim_{x\to\infty} f(x) = \lim_{y\to0^+} f(y^{-1}), \quad \lim_{x\to-\infty} f(x) = \lim_{y\to0^-} f(y^{-1}),$$

where the equality means that either both limits exist and are equal or none of the limits exist.

Proof. We prove the first equality, while the second one can be similarly shown. Let $L \in \mathbb{R}$. Note that $f(x) \to L$ as $x \to \infty$ if

$$\forall\epsilon > 0, \quad \exists M > 0 \quad \text{s.t.} \quad |f(x) - L| < \epsilon, \quad \forall x > M.$$

By setting $y = 1/x$ and $\delta = 1/M$, the above statement is equivalent to

$$\forall\epsilon > 0, \quad \exists\delta > 0 \quad \text{s.t.} \quad |f(1/y) - L| < \epsilon, \quad \forall 0 < y < \delta.$$

$\square$

Example 2.19. Consider the function $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$. Note that $f(1/y) = g(y) := \frac{\sqrt{2+y^2}}{3-5y}$. By Lemma 2.12 and the continuity of $g(y)$ at 0, we have $f(x) \to g(0) = \frac{\sqrt{2}}{3}$ as $x \to \infty$.

$\boxed{\textbf{Limit laws for limits at infinity}}$ Assume that the limits of f, g at infinity exist. Then,

$$\lim_{x\to\infty}(f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x), \quad \lim_{x\to\infty} cf(x) = c\lim_{x\to\infty} f(x),$$

and

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} f(x) \times \lim_{x\to\infty} g(x), \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)},$$

where the latter requires $\lim_{x\to\infty} g(x) \neq 0$.

Power law and root law for limits at infinity Assume that the limit of f at infinity exists. Then,

$$\lim_{x\to\infty}[f(x)]^n = \left[\lim_{x\to\infty} f(x)\right]^n \quad \lim_{x\to\infty}[f(x)]^{1/n} = \left[\lim_{x\to\infty} f(x)\right]^{1/n},$$

where the second identity requires $f(x) \geq 0$ for x large enough when n is even.

Theorem 2.13 (The Squeeze theorem for limits at infinity)**.** *Assume that there is $M > 0$ such that $f(x) \leq g(x) \leq h(x)$ for $x \geq M$. If the limits of f, h at infinity exist and equal L, then the limit of g at infinity equals L.*

Remark 2.21. The above limit laws, power and root laws and squeeze theorem also apply for the limits at $-\infty$.

Definition 2.13 (Infinite limits at infinity)**.** We write

$$\lim_{x\to\infty} f(x) = \infty \quad (\text{resp.} \ -\infty),$$

if the value of $f(x)$ can be arbitrarily large (resp. arbitrarily large negative) as x is sufficiently large, or equivalently

$$\forall M > 0, \quad \exists N > 0, \quad \text{s.t.} \quad f(x) > M \quad (\text{resp. } f(x) < -M), \quad \forall x > N.$$

Remark 2.22. Lemma 2.12 also applies for infinite limits at infinity.

Example 2.20. Let $f(x) = x^2 - x$ and $g(x) = \frac{x^2}{x+3}$. Note that the limit laws do not apply for f, g at infinity because they produce the indeterminate forms of $\infty - \infty$ and ∞/∞. To see a precise estimation, we write

$$f(x) = x(x-1) \geq x, \quad \forall x \geq 2, \quad g(x) = \frac{x}{1 + 3/x} \geq \frac{x}{2}, \quad \forall x \geq 3,$$

which imply

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty.$$