2.6. **Derivatives and rates of changes.** (Sec. 2.7 in the textbook)

Definition 2.14. The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is defined to be a straight line passing through $(a, f(a))$ with slope

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided such a limit exists.

Definition 2.15. The derivative of f at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

Remark 2.23. If $f'(a)$ exists, then the tangent line to the curve $y = f(x)$ at $(a, f(a))$ is given by $y = f(a) + (x - a)f'(a)$.

Example 2.21. Let $f(x) = x^2 + 1$. The derivative of f at a is given by

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(x^2 + 1) - (a^2 + 1)}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

The tangent line to the curve $y = f(x)$ at $(a, a^2 + 1)$ is $y = 2a(x - a) + (a^2 + 1)$.

To see an interpretation of the derivative, define

$$\Delta x = x - a : \text{the increment of } x, \quad \Delta y = f(x) - f(a) : \text{the corresponding change in } y,$$

and

$$\Delta y / \Delta x : \text{the average rate of change of } y \text{ with respect to } x \text{ over the interval } [a, x].$$

The limit of $\Delta y/\Delta x$ is called the instantaneous rate of change of y w.r.t. x at $x = a$.

Practically, let's consider an object moving on the real line with equation of motion $s = f(t)$, where t is the time and s is the position on $\mathbb{R}$. For $t > a$, set $\Delta t = t - a$ and $\Delta s = f(t) - f(a)$. Then, $\Delta s/\Delta t$ is the average speed of this object during the time interval $[a, x]$. As t approaches a, the limit

$$\lim_{t \to a} \frac{\Delta s}{\Delta t} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} = f'(a)$$

is known as the instantaneous speed of this object at time $t = a$. Roughly speaking, if t is close to a, then

$$f(t) - f(a) \approx f'(a)(t - a) \quad \text{or} \quad f(t) \approx f(a) + f'(a)(t - a),$$

where $u \approx v$ means that u is approximately (not precisely) equal to v. For an illustration of the above discussion, let $f(t) = 3t^2 - 2t + 1$ and $a = 2$. One may compute $f(2) = 9$ and $f'(2) = 10$, which lead to

$$f(2.001) \approx f(2) + f'(2) \times 0.001 = 9.01.$$

In fact, $f(2.001) = 9.010003$.

Example 2.22. Let $f(x) = x \sin(1/x)$ for $x \neq 0$, $f(0) = 0$ and $g(x) = xf(x)$. Note that

$$\frac{f(x) - f(0)}{x - 0} = \sin(1/x), \quad \frac{g(x) - g(0)}{x - 0} = x \sin(1/x) = f(x).$$

As it has been shown before that $\sin(1/x)$ has no limit at 0, $f'(0)$ does not exist. For $g'(0)$, consider the inequality $-|x| \leq f(x) \leq |x|$ for $x \in \mathbb{R}$. By the squeeze theorem, f is continuous at 0. This implies $g'(0) = f(0) = 0$.