2.7. **The derivative as a function.** (Sec. 2.8 in the textbook)

Definition 2.16. The derivative of f is denoted and defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

The domain of f' is the set of points x such that the above limit exists, which must be a subset of the domain of f.

Remark 2.25. Note that $f'(x)$ is the slope of the tangent line to the curve $y = f(x)$ at $(x, f(x))$.

Example 2.23. Let $f(x) = (x-1)/(x+1)$. Clearly, the domain of f is $(-\infty,-1)\cup(-1,\infty)$ and $f(x) = 1 - \frac{2}{x+1}$. This implies that, for all $x \neq 1$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{-2}{x+h+1} - \frac{-2}{x+1}\right) \\ &= \lim_{h\to 0} \frac{2}{(x+1)(x+h+1)} = \frac{2}{(x+1)^2}. \end{aligned}$$

Definition 2.17. A function f is called differentiable at a if $f'(a)$ exists and called differentiable on (a, b) if it is differentiable at every point in (a, b).

Example 2.24. Consider the following function,

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 1 \\ 2x^2 - 1 & \text{if } x < 1 \end{cases}.$$

One may show that f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$ except at 1, since

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = 2, \quad \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = 4.$$

Remark 2.26. When writing $y = f(x)$, we also use $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f(x)$ and $Df(x)$ to denote $f'(x)$.

Theorem 2.14. *If f is differentiable at a, then f is continuous at a.*

Proof. The proof follows immediately from limit laws and

$$\lim_{x\to a}[f(x) - f(a)] = \left(\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right) = 0.$$

□

Remark 2.27. The converse of Theorem 2.14 can fail. In general, f is not differentiable at a if f is not continuous at a or f is continuous at a but its graph has a "corner" at $(a, f(a))$.

The derivative of f', i.e. $(f')'$, is called the second derivative of f and written as f''. Inductively, the derivative of the nth derivative of f is called the $(n+1)$st derivative of f. The nth derivative of f is written as $\frac{d^n y}{dx^n}$ or $f^{(n)}$.

Example 2.25. For $f(x) = x^3 - 2x$, one may derive by following the definition of derivatives that $f'(x) = 3x^2 - 2$, $f''(x) = 6x$, $f'''(x) = 6$ and $f^{(n)}(x) = 0$ for $n \geq 4$.