3.1. **Derivatives of polynomials and exponential functions.** (Sec. 3.1 in the textbook)

Theorem 3.1. *For $n \in \mathbb{N}$ and $c \in \mathbb{R}$, one has*

$$\frac{d}{dx}(c) = 0, \quad \frac{d}{dx}(x^n) = nx^{n-1}, \quad \frac{d}{dx}(x^{-n}) = (-n)x^{-n-1} \quad \forall x \neq 0.$$

Proof. The first one is clear. For the second one, note that

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}).$$

This implies

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) = na^{n-1}.$$

Immediately, for $a \neq 0$,

$$\lim_{x\to a} \frac{x^{-n} - a^{-n}}{x-a} = \lim_{x\to a}\left(\frac{-1}{x^n a^n} \times \frac{x^n - a^n}{x-a}\right) = \frac{-1}{a^{2n}} \times (na^{n-1}) = (-n)a^{-n-1}.$$

□

Remark 3.1. In fact, for $r \in \mathbb{R}$, $\frac{d}{dx}(x^r) = rx^{r-1}$ for $x > 0$.

Theorem 3.2. *Let c be a constant. If f, g are differentiable at a, then $f+g, cf$ are differentiable at a with $(f+g)'(a) = f'(a) + g'(a)$ and $(cf)'(a) = cf'(a)$. In particular, if f, g are differentiable on an open interval I, then $(f+g)' = f' + g'$ and $(cf)' = cf'$ on I.*

Proof. The differentiation rule for addition is obtained by

$$\lim_{x\to a} \frac{[f(x)+g(x)] - [f(a)+g(a)]}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a) + g'(a),$$

while that for scalar multiplication can be proved in a similar way and omitted. □

Corollary 3.3. *For $P(x) = \sum_{k=0}^n a_k x^k$, $P'(x) = \sum_{k=1}^n k a_k x^{k-1}$.*

To see derivatives of exponential functions, let $f(x) = b^x$ with $b > 0$. Note that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h\to 0} \frac{b^h - 1}{h}.$$

If f is differentiable at 0, then it is differentiable on $\mathbb{R}$ and $f'(x) = f(x)f'(0)$. To see the existence of $f'(0)$, we set $a_n = n(b^{1/n} - 1)$ and $g(y) = n(y^{n+1}-1) - (n+1)(y^n - 1)$. It is easy to show that $g'(y) > 0$ for $y > 1$. By the increasing/decreasing test(see Chapter 4), $g(y) > g(1) = 0$ for $y > 1$. For the case of $b > 1$, replacing y with $b^{1/[n(n+1)]}$ yields $a_n > a_{n+1}$. As $a_n > 0$ for all n, a_n converges. Set $L = \lim_n a_n$. Note that $[n/(n+1)]a_{n+1} < [f(x) - f(0)]/x < [(n+1)/n]a_n$ for $1/(n+1) < x < 1/n$. By the squeeze theorem, $[f(x) - f(0)]/x \to L$ as $x \to 0^+$. By the continuity of f at 0, the left limit of $[f(x) - f(0)]/x$ can be derived from and equal to its right limit. This implies that f is differentiable at 0. For the case of $0 < b < 1$, one may prove the differentiability with the equality $\frac{b^x - 1}{x} = (-b^x)\frac{(1/b)^x - 1}{x}$.

Recall that e is the number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

From the above definition, one may compute

$$\lim_{h\to 0} \frac{b^h - 1}{h} = \lim_{h\to 0} \frac{e^{h\ln b} - 1}{h} = \lim_{k\to 0}\left(\frac{e^k - 1}{k}\right)\ln b = \ln b, \quad \forall b > 0.$$

Immediately, this implies $(b^x)' = (\ln b)b^x$ and $(e^x)' = e^x$.