3.2. **The product and quotient rules.** (Sec. 3.2 in the textbook)

Theorem 3.4. *Let f, g be differentiable at a. Then, fg is differentiable at a and*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Proof. Since f and g are differentiable at a, they are continuous at a. Note that

$$f(a+h)g(a+h) - f(a)g(a) = [f(a+h) - f(a)]g(a+h) + [g(a+h) - g(a)]f(a).$$

This desired identity is then given by

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} \times g(a+h)\right] = f'(a)g(a), \quad \lim_{h\to 0}\left[\frac{g(a+h)-g(a)}{h} \times f(a)\right] = g'(a)f(a).$$

□

Example 3.1. For $f(x) = xe^x$, $f'(x) = \frac{d}{dx}(x) \cdot e^x + x\frac{d}{dx}(e^x) = (x+1)e^x$.

Example 3.2. Let $f(x) = \sqrt{x}g(x)$. If $g'(a)$ exists for some $a > 0$, then $f'(a) = \frac{g(a)}{2\sqrt{a}} + \sqrt{a}g'(a)$. When $g(2) = 1$ and $g'(2) = -1$, the replacement of $a = 2$ yields $f'(2) = -\frac{3\sqrt{2}}{4}$.

Theorem 3.5. *Let f, g be functions differentiable at a with $g(a) \neq 0$. Then, f/g is differentiable at a and*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.$$

Proof. Since g is differentiable at a, g is continuous at a. As a result of $g(a) \neq 0$, we know that $g(x) \neq 0$ for x sufficiently close to a. This implies f/g is well-defined in a neighborhood of a. When $f(x) = 1$ for all x, one has

$$\left(\frac{1}{g}\right)'(a) = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{g(a+h)} - \frac{1}{g(a)}\right] = \lim_{h\to 0}\frac{-1}{g(a)g(a+h)} \times \frac{g(a+h)-g(a)}{h} = \frac{-g'(a)}{(g(a))^2}.$$

In addition with the product rule, we obtain

$$\left(\frac{f}{g}\right)'(a) = f'(a) \times \frac{1}{g(a)} + f(a) \times \frac{-g'(a)}{g^2(a)} = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

□

Example 3.3. Let $f(x) = \frac{2x^3 - x + 1}{e^x + 1}$. To see $f'(0)$, we first compute

$$\begin{aligned} f'(x) &= \frac{1}{(e^x+1)^2}\left[(e^x+1)\frac{d}{dx}(2x^3 - x + 1) - (2x^3 - x + 1)\frac{d}{dx}(e^x + 1)\right] \\ &= \frac{(e^x+1)(6x^2-1) - (2x^3 - x + 1)e^x}{(e^x+1)^2}. \end{aligned}$$

Letting $x = 0$ yields $f'(0) = -\frac{3}{4}$.

Example 3.4. To see the derivative of e^{-x}, we write $e^{-x} = \frac{1}{e^x}$ and compute

$$\frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{1}{e^{2x}}\left[e^x\frac{d}{dx}(1) - \frac{d}{dx}(e^x)\right] = -e^{-x}.$$