3.3. **Derivatives of trigonometric functions.** (Sec. 3.3 in the textbook)

First, recall the following formulas.

$$\sin(x+h) = \sin x \cos h + \cos x \sin h, \quad \cos(x+h) = \cos x \cos h - \sin x \sin h.$$

This implies

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = (\sin x) \lim_{h\to 0} \frac{\cos h - 1}{h} + (\cos x) \lim_{h\to 0} \frac{\sin h}{h} \tag{3.1}$$

and

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = (\cos x) \lim_{h\to 0} \frac{\cos h - 1}{h} - (\sin x) \lim_{h\to 0} \frac{\sin h}{h}, \tag{3.2}$$

provided that the derivatives of sine and cosine functions at 0 exists.

Lemma 3.6. *For any $\theta \in (0, \pi/2)$, $\sin\theta < \theta < \tan\theta$.*

Proof. A geometric proof of this lemma is given in the textbook and omitted. □

By Lemma 3.6, $\cos\theta < \sin\theta/\theta < 1$ for $\theta \in (0, \pi/2)$. Since the limit of $\cos x$ at 0 equals 1, the squeeze theorem implies

$$\lim_{h\to 0^+} \frac{\sin h}{h} = 1, \quad \lim_{h\to 0^-} \frac{\sin h}{h} = \lim_{k\to 0^+} \frac{\sin(-k)}{-k} = \lim_{k\to 0^+} \frac{\sin k}{k} = 1.$$

Hence, $\sin h/h \to 1$ as $h \to 0$. For the cosine function, note that

$$\cos\theta - 1 = \frac{\cos^2\theta - 1}{\cos\theta + 1} = \frac{-\sin^2\theta}{\cos\theta + 1}, \quad \forall \theta \in (-\pi/2, \pi/2).$$

By the limit laws, this implies

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \left(\frac{\sin h}{h} \times \frac{-\sin h}{\cos h + 1} \right) = 0.$$

As a consequence, we obtain

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \quad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

The derivatives of sine and cosine functions are given by (3.1) and (3.2), while the derivatives of other trigonometric functions can be derived using the product and quotient rules.

$$\frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x, \quad \frac{d}{dx} \tan x = \sec^2 x,$$

and

$$\frac{d}{dx} \cot x = -\csc^2 x, \quad \frac{d}{dx} \sec x = \tan x \sec x, \quad \frac{d}{dx} \csc x = -\cot x \csc x.$$

Example 3.5. For $f(x) = \frac{\sec x}{1+\tan x}$, $f'(x) = \frac{(\tan x - 1)\sec x}{(1+\tan x)^2}$.

Example 3.6. To compute the limit, say L, of $\frac{\sin(3x)}{2x}$ at 0, let $y = 3x$. Clearly, $y \to 0$ if and only if $x \to 0$. This implies

$$L = \lim_{y\to 0} \frac{\sin y}{(2y)/3} = \frac{3}{2} \lim_{y\to 0} \frac{\sin y}{y} = \frac{3}{2}.$$