3.5. **Implicit differentiation.** (Sec. 3.5 in the textbook)

Consider the well-known *folium of Decartes*, which is the set, say S, of solutions (x, y) to $x^3 + y^3 = 6xy$. There are many ways to express part of S as a function of x. Suppose that $y = f(x)$ for $x \in I$ solves $x^3 + y^3 = 6xy$, where I is an open interval. Clearly, one has $x^3 + [f(x)]^3 = 6xf(x)$ for $x \in I$. If f is differentiable at x, then

$$3x^2 + 3[f(x)]^2 f'(x) = 6[f(x) + xf'(x)].$$

This implies

$$f'(x) = \frac{x^2 - 2f(x)}{2x - [f(x)]^2} = \frac{x^2 - 2y}{2x - y^2}.$$

Using this formula, one can see that the tangent line to the folium of Decartes at $(3, 3)$ has slope -1 if any. To see the points in the first quadrant with horizontal tangent lines, we assume that $f'(x) = 0$. Again, the above formula implies $x^2 = 2y$ and $2x \neq y^2$. Along with the identity $x^3 + y^3 = 6xy$, we obtain

$$6xy = x^3 + y^3 = 2xy + y^3 \quad \Rightarrow \quad 4x = y^2 = \frac{x^4}{4} \quad \Rightarrow \quad x = 2^{4/3},\ y = 2^{5/3}.$$

Clearly, $2 \times 2^{4/3} \neq (2^{5/3})^2$ and, thus, $(2^{4/3}, 2^{5/3})$ is the unique solution to $f'(x) = 0$ in the first quadrant.

Remark 3.3. In the above setting, f is call an implicit function and we refer the reader to the *implicit function theorem* for the differentiability of f.

Finding the derivative of an implicit function to the equation $F(x, y) = 0$

- **Step 1:** Regard y as a function of x.
- **Step 2:** Differentiate both sides of $F = 0$. w.r.t. x.
- **Step 3:** Express $\frac{dy}{dx}$ as a function of x and y.
- Always remember that the pair (x, y) in Step 3 is a solution to $F(x, y) = 0$.

Example 3.10. Consider the curve $x^2 + y^2 = 9$ and regard y as a function of x. Differentiating both sides of $x^2 + y^2 = 9$ yields $2x + 2y\frac{dy}{dx} = 0$. This implies $\frac{dy}{dx} = -\frac{x}{y}$ for $y \neq 0$. At the point $(1, 2\sqrt{2})$, the tangent line has slope $\frac{-1}{2\sqrt{2}}$ and formulated by $y = \frac{\sqrt{2}}{4}(9 - x)$.

Theorem 3.8. *Let f be a one-to-one function defined on an open interval I. Assume that f is continuously differentiable on I and $f'(a) \neq 0$. Then, f^{-1} is differentiable at $f(a)$ and*

$$\frac{df^{-1}}{dx}(f(a)) = \frac{1}{f'(a)}.$$

Proof. The differentiability of f^{-1} at $f(a)$ is given by the inverse function theorem and omitted. To see its value, note that $x = f(f^{-1}(x))$ for $x \in f(I)$. By the chain rule, if f^{-1} is differentiable at x, then

$$1 = \frac{dx}{dx} = \frac{d}{dx} f \circ f^{-1}(x) = f'(f^{-1}(x))(f^{-1})'(x).$$

Further, if $f'(f^{-1}(x)) \neq 0$, then $(f^{-1})'(x) = 1/f'(f^{-1}(x))$. The desired identity is obtained by letting $x = f(a)$. □

Example 3.11. Let $y = f(x) = \sin x$. Then, $f'(x) = \cos x$ for $|x| < \pi/2$. This implies

$$\frac{d}{dy}(\sin^{-1} y) = \frac{1}{\cos x} = \frac{1}{\cos(\sin^{-1}(y))}, \quad \forall y \in (-1, 1).$$

Note that, for $|y| \leq 1$,

$$\cos(\sin^{-1} y) = \sqrt{1 - \sin^2(\sin^{-1} y)} = \sqrt{1 - y^2}.$$

Hence, we have

$$\frac{d}{dy}(\sin^{-1} y) = \frac{1}{\sqrt{1 - y^2}}, \quad \forall -1 < y < 1.$$

Next, let $y = g(x) = \tan x$. Clearly, $g'(x) = \sec^2 x = 1 + \tan^2 x$. For $-\pi/2 < x < \pi/2$, one may compute

$$\frac{d}{dy}(\tan^{-1} y) = \frac{1}{1 + \tan^2(\tan^{-1} y)} = \frac{1}{1 + y^2}.$$

One may use similar computations to derive

$$\frac{d}{dy}(\cos^{-1} y) = -\frac{1}{\sqrt{1 - y^2}}, \quad \frac{d}{dy}(\cot^{-1} y) = -\frac{1}{1 + y^2}$$

and

$$\frac{d}{dy}(\sec^{-1} y) = \frac{1}{y\sqrt{y^2 - 1}}, \quad \frac{d}{dy}(\csc^{-1} y) = -\frac{1}{y\sqrt{y^2 - 1}}.$$

Example 3.12. For $f(x) = \sin^{-1}(\sqrt{1 - x^2})$ with $0 < |x| \leq 1$, $f'(x) = \frac{-x}{|x|\sqrt{1-x^2}}$.