3.6. **Derivatives of logarithmic functions.** (Sec. 3.6 in the textbook)

Theorem 3.9. *For* $b > 0$ *and* $f(x) = \log_b x$, $f'(x) = \frac{1}{x \ln b}$. *In particular,* $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Proof. Let $y = f(x) = \log_b x$ and $x = g(y) = b^y$. Then,

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{(\ln b)b^{\log_b x}} = \frac{1}{(\ln b)x}.$$

□

Example 3.13. Let $f(x) = \ln|x|$ for $x \neq 0$. When $x > 0$, $f'(x) = x^{-1}$. When $x < 0$,

$$f'(x) = \frac{d}{dx}\ln(-x) = \frac{1}{-x} \times (-1) = \frac{1}{x}.$$

This implies $f'(x) = x^{-1}$ for $x \neq 0$.

Example 3.14. Let $g(x)$ be a function taking values on $(0, \infty)$ and set $f(x) = \ln g(x)$. By the chain rule, one has $f'(x) = \frac{g'(x)}{g(x)}$. In particular, when $g(x) = \sin x$, we have $\frac{d}{dx}\ln(\sin x) = \frac{\cos x}{\sin x} = \cot x$ for $0 < x < \pi$.

Logarithmic differentiation Let $f(x) = f_1(x) \times \cdots \times f_n(x)$. To compute $f'(x)$ or equivalently $\frac{dy}{dx}$, we consider the following strategy.

- **Step 1:** Write $y = f(x)$ and derive $\ln y = \ln f(x) = \ln f_1(x) + \cdots + \ln f_n(x)$.
- **Step 2:** Differentiate both sides w.r.t. x, while $\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$.

Example 3.15. Let $y = f(x) = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$. Using the above strategy, we have

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2).$$

This implies

$$\frac{f'(x)}{f(x)} = \frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2},$$

and, hence,

$$f'(x) = f(x) \times \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) \quad \forall x \in (0, \infty).$$

Remark 3.4. Let $a, b \in \mathbb{R}$ and f, g be differentiable functions. Then,

$$\frac{d}{dx}(a^b) = 0, \quad \frac{d}{dx}(f(x)^b) = bf(x)^{b-1}f'(x), \quad \frac{d}{dx}(a^{g(x)}) = (\ln a)a^{g(x)}g'(x),$$

and

$$\frac{d}{dx}(f(x)^{g(x)}) = \frac{d}{dx}(e^{g(x)\ln f(x)}) = f(x)^{g(x)}\left(g'(x)\ln f(x) + \frac{f'(x)g(x)}{f(x)}\right).$$

In the case that $f(x) = g(x) = x$, we have $\frac{d}{dx}(x^x) = x^x(\ln x + 1)$.

Remark 3.5. Let $f(x) = \ln x$ and $c \in \mathbb{R}$. Since $f'(1) = 1$, we have

$$c = c\lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h/c} = \lim_{k\to 0}\ln(1+ck)^{1/k}.$$

Since $g(x) = e^x$ is continuous, this implies

$$e^c = g(c) = g\left(\lim_{k\to 0}\ln(1+ck)^{1/k}\right) = \lim_{k\to 0}g\left(\ln(1+ck)^{1/k}\right) = \lim_{k\to 0}(1+ck)^{1/k}.$$