3.7. **Related rates.** (Sec. 3.9 in the textbook)

Example 3.16. Air is pumped into a spherical balloon with rate 100 cm^3/s. To see how fast the radius of the balloon increasing, let $V(r)$ be the volume of the balloon with radius r. Note that $V(r) = \frac{4}{3}\pi r^3$. By the chain rule, one has

$$100 = \frac{dV(r)}{dt} = \frac{dV(r)}{dr} \cdot \frac{dr}{dt} = 4\pi r^2(t)r'(t),$$

which implies $r'(t) = 25/[\pi r^2(t)]\ cm/s$.

Example 3.17. Consider a ladder of 10 meter long rested against a vertical wall. When the bottom of the ladder slides away, we are interested in the rate that the top slides down the wall. Suppose that the bottom and the top of the ladder is x and y meters from the wall and the floor. Clearly, $x^2 + y^2 = 100$. By regarding x and y as functions of time t, one may compute

$$x(t)x'(t) + y(t)y'(t) = 0 \quad \Rightarrow \quad \frac{y'(t)}{x'(t)} = -\frac{x(t)}{y(t)} \quad \text{or} \quad y'(t) = \frac{-x(t)x'(t)}{\sqrt{100 - x^2(t)}}.$$

3.8. **Linear approximations and differentials.** (Sec. 3.10 in the textbook)

Definition 3.2. Let f be a function differentiable at a. The approximation $f(x) \approx f(a) + (x-a)f'(a)$ is called the *linear approximation* or *tangent line approximation* of f at a. The *linearization* of f at a refers to the function $L(x) = f(a) + (x-a)f'(a)$.

Example 3.18. For $f(x) = \sqrt{x+3}$, one has $f'(x) = \frac{1}{2\sqrt{x+3}}$. Immediately, $f'(1) = 1/4$ and the linearization of f at 1 is given by $L(x) = \frac{x+7}{4}$. As $f(x) \approx L(x)$ as x is close to 1, we have

$$\sqrt{3.98} = f(0.98) \approx L(0.98) = 1.995, \quad \sqrt{4.01} = f(1.01) \approx L(1.01) = 2.0025.$$

In fact, $\sqrt{3.98} = 1.994993...$ and $\sqrt{4.01} = 2.002498....$

Definition 3.3. Let f be a differentiable function and write $y = f(x)$. The *differential* dx is an independent variable and the *differential* dy is defined to be a function depending on dx through the equation $dy = f'(x)dx$.

Remark 3.6. If Δx is the increment of x, then $\Delta y = f(x + \Delta x) - f(x) \approx dy$ when $\Delta x = dx$.

Example 3.19. For $y = f(x) = x^2 + 3x - 5$, one has $dy = f'(x)dx = (2x+3)dx$. When $x = 0$ and $\Delta x = dx = 0.01$,

$$dy = 3dx = 0.03, \quad \Delta y = f(\Delta x) - f(0) = \Delta x(3 + \Delta x) = 0.0301.$$

Example 3.20. Consider a balloon with measured radius 20 cm and possible error 0.01 cm. To see the error on the computation of the volume of the balloon, let $V(r)$ be the volume of a sphere with radius r. Since $V(r) = \frac{4}{3}\pi r^3$, we have $dV = 4\pi r^2 dr$. When $r = 20$ and $|dr| \le 0.01$,

$$|dV| \le 4\pi 20^2 \cdot 0.01 = 16\pi(cm^3),$$

while the relative error is

$$\left|\frac{\Delta V}{V}\right| \approx \left|\frac{dV}{V}\right| = \frac{3|dr|}{r} \le 0.0015.$$