4.1. **Maximum and minimum values.** (Sec. 4.1 in the textbook)

Definition 4.1. Let f be a function with domain D and $c \in D$.

(1) f has an *absolute maximum* or *global maximum* at c if $f(x) \leq f(c)$ for $x \in D$. In this case, $f(c)$ is called the *maximum value* of f on D.
(2) f has an *absolute minimum* or a *global minimum* at c if $f(x) \geq f(c)$ for $x \in D$. In this case, $f(c)$ is called the *minimum value* of f on D.
(3) f has an *absolute extremum* or a *global extremum* at c if f has a global maximum at c or a global minimum at c. The value $f(c)$ is called an *extremum value* of f.

Example 4.1. Let f be functions defined by $f(x) = x^2$. Then,

- When $D = [-1, 1]$, f has absolute maxima at ± 1 and an absolute minimum at 0.
- When $D = (-1, 1)$, f has no absolute maximum but has an absolute minimum at 0.
- When $D = (-1, 0) \cup (0, 1)$, f has no absolute extremum.

Theorem 4.1 (The extremum value theorem)**.** *Let f be a continuous function defined on a closed interval $[a, b]$. Then, there exist $c, d \in [a, b]$ such that f has an absolute maximum and an absolute minimum at c and d.*

Proof. Consider the following two cases.

Case 1: f is bounded on $[a, b]$, i.e. there is $M > 0$ such that $|f(x)| \leq M$ for $a \leq x \leq b$. In this case, we may assume the existence of constants $\ell < L$ such that $\ell \leq f(x) \leq L$ for $a \leq x \leq b$ and, for any $n \in \mathbb{N}$, there are $x_n, y_n \in [a, b]$ such that $\ell \leq f(x_n) < \ell + 1/n$ and $L - 1/n < f(y_n) \leq L$. By the dichotomy method, one may select convergent subsequences, $(x_{n_k})_{k=1}^{\infty}$ and $(y_{m_k})_{k=1}^{\infty}$, with limit $x, y \in [a, b]$. By the continuity of f and the squeeze, we obtain

$$f(x) = \lim_{k \to \infty} f(x_{n_k}) = \ell, \quad f(y) = \lim_{k \to \infty} f(y_{m_k}) = L.$$

Case 2: f is unbounded on $[a, b]$, i.e. there is a sequence $(x_n)_{n=1}^{\infty}$ such that such that $f(x_n) \to \infty$ or $f(x_n) \to -\infty$. In either case, one may use the dichotomy method to select a convergent subsequence $(x_{n'_k})_{k=1}^{\infty}$ with limit x. By the continuity of f, this implies

$$f(x) = \lim_{k \to \infty} f(x_{n'_k}) = \infty \quad \text{or} \quad f(x) = \lim_{k \to \infty} (x_{n'_k}) = -\infty,$$

which is obviously a contradiction. □

Definition 4.2. f has a *local maximum* or *relative maximum* at c if there exists $\delta > 0$ such that $f(x) \leq f(c)$ for $|x - c| < \delta$. f has a *local minimum* or *relative minimum* at c if there is $\delta > 0$ such that $f(x) \geq f(c)$ for $|x - c| < \delta$. Either case is called a *local extremum* of f.

Remark 4.1. f has a global (resp. local) maximum at c if and only if $-f$ has a global (resp. local) minimum at c.

Remark 4.2. If f has a global extremum at c and f is defined in a neighborhood of c, then f has a local extremum at c.

Example 4.2. For $x \in \mathbb{R}$, let $f(x) = \sin x$ and $g(x) = x^3$. Then, f has global (local) maximum at $2n\pi$ for $n \in \mathbb{Z}$ with maximum value 1 and global (local) minimum at $(2n + 1)\pi$ for $n \in \mathbb{Z}$ with minumum value -1. For g, there is no global (local) extremum at all.

Theorem 4.2 (Fermat's theorem)**.** *If f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$.*

Proof. By Remark 4.1, it suffices to prove the case of local maximum. Suppose f has a local maximum at c and choose $\delta > 0$ such that $f(x) \leq f(c)$ for $|x-c| < \delta$. This implies

$$\frac{f(x)-f(c)}{x-c} \leq 0, \quad \forall c < x < c+\delta, \quad \frac{f(x)-f(c)}{x-c} \geq 0 \quad \forall c-\delta < x < c.$$

Since $f'(c)$ exists, we have

$$f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \leq 0, \quad f'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \geq 0.$$

Hence, $f'(c) = 0$. ☐

Remark 4.3. By Fermat's theorem, if f is differentiable, then $f' = 0$ is necessary for the existence of local extremum. However, the inverse statement is not necessarily true. See e.g. $f(x) = x^3$.

Definition 4.3. A *critical number* or *critical point* of a function f is a number c such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 4.3. *If f has a local extremum at c, then c is a critical number of f.*

Remark 4.4. The inverse statement of Theorem 4.3 is not necessarily true.

The closed interval method Let f be a function defined on $[a,b]$. The following is a scheme of finding the extremum values of f on $[a,b]$.

- **Step 1:** Compute $f(a)$, $f(b)$ and those values of f at all critical numbers in (a,b).
- **Step 2:** The largest and smallest values in Step 1 are the extremum values of f.

Example 4.3. Let $f(x) = x^3 - 6x^2 + 9x - 1$ for $x \in [0,4]$. Note that $f'(x) = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$. Clearly, 1 and 3 are solutions to $f'(x) = 0$ with $x \in (0,4)$. By the closed interval method, the extremum values of f on $[0,4]$ are

$$\max\{f(0), f(1), f(3), f(4)\} = 3, \quad \min\{f(0), f(1), f(3), f(4)\} = -1.$$