4.2. **The mean value theorem.** (Sec. 4.2 in the textbook)

Theorem 4.4 (Rolle's theorem)**.** *Let f be a function defined on $[a,b]$. Suppose f is continuous on $[a,b]$, differentiable on (a,b) and $f(a)=f(b)$. Then, there is $c\in(a,b)$ such that $f'(c)=0$.*

Proof. Obviously, this theorem holds for constant functions on $[a,b]$. For other cases, it loses no generality to assume that there is $x_0\in(a,b)$ such that $f(x_0)>f(a)$. Since f is continuous on $[a,b]$, the extremum value theorem implies the existence of $c\in[a,b]$ such that f has its global maximum at $c\in[a,b]$. As $f(x_0)>f(a)=f(b)$, it must be the case that $c\in(a,b)$. By the differentiability of f on (a,b), Fermat's theorem implies $f'(c)=0$. □

Example 4.4. Consider the roots of $f(x)=x^5+2x-2$. As f is a polynomial, it is continuous and differentiable on $\mathbb{R}$. Note that $f(1)=1>0$ and $f(0)=-2<0$. By the intermediate value theorem, there is $c\in(0,1)$ such that $f(c)=0$. Observe that $f'(x)=5x^4+2>0$ for $x\in\mathbb{R}$. By Rolle's theorem, if $d\neq c$ is another root of f, then there is a constant x_0 between c and d such that $f'(x_0)=0$, which contradicts the fact $f'>0$. This implies that c is the unique root of f.

Theorem 4.5 (The mean value theorem)**.** *Let f be a function on $[a,b]$. Assume that f is continuous on $[a,b]$ and differentiable on (a,b). Then, there is $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.*

Example 4.5. Suppose f is a differentiable function on $\mathbb{R}$ with $f(0)=-1$ and $f'(x)\geq 2$ for all x. To examine how large $f(10)$ can be, we may use the mean value theorem to compute $\frac{f(10)-f(0)}{10-0}=f'(c)$ for some $c\in(0,10)$. This implies $f(10)=f(0)+10f'(c)\geq -1+10\cdot 2=19$.

Remark 4.5. The mean value theorem can fail without the differentiability of f on (a,b) (see e.g. $f(x)=|x|$ with $a=-1, b=1$) or the continuity of f at a,b (see e.g. $f(x)=x$ for $x\in(-1,1]$ and $f(-1)=1$).

Corollary 4.6. *If $f'(x)=0$ for $x\in(a,b)$, then f is a constant function. In particular, if g,h are differentiable on (a,b) and $g'=h'$, then there is $c\in\mathbb{R}$ such that $g=h+c$ on (a,b).*

Proof. The second part is a simple corollary of the first part through the identity $f=g-h$. For the first part, assume that $f'=0$ on (a,b). Fix $x_0\in(a,b)$. When $x\in(a,x_0)$, one may apply the mean value theorem to find $y\in(x,x_0)$ such that $f(x_0)-f(x)=f'(y)(x_0-x)$ or equivalently $f(x)=f(x_0)$. Similarly, we can show that $f(x)=f(x_0)$ for $x\in(x_0,b)$. □

Example 4.6. Let $f(x)=\sin^{-1}x+\cos^{-1}x$ for $x\in[-1,1]$. Note that

$$f\left(\frac{1}{\sqrt{2}}\right)=\sin^{-1}\frac{1}{\sqrt{2}}+\cos^{-1}\frac{1}{\sqrt{2}}=\frac{\pi}{2},\quad f'(x)=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}=0,\quad \forall x\in(-1,1).$$

By Corollary 4.6, $f(x)=\pi/2$ for $x\in(-1,1)$. Since f is continuous on $[-1,1]$, we have

$$f(-1)=\lim_{x\to(-1)^+}f(x)=\frac{\pi}{2},\quad f(1)=\lim_{x\to 1^-}f(x)=\frac{\pi}{2}.$$

Thus, $f=\pi/2$ on $[-1,1]$.