4.3. **How derivatives affect the shape of a graph.** (Sec. 4.3 in the textbook.)

Recall that a function f is increasing on (a, b) if $f(x) < f(y)$ for $a < x < y < b$ and decreasing on (a, b) if $f(x) > f(y)$ for $a < x < y < b$.

Theorem 4.7 (Increasing/Decreasing test)**.** *Let f be a function defined on (a, b).*

(1) *If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b).*
(2) *If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).*

Proof. It loses no generality to assume that $f' > 0$ on (a, b). Let $a < x < y < b$. By the mean value theorem, there is $z \in (x, y)$ such that $f(y) - f(x) = f'(z)(y - x) > 0$, which implies $f(y) > f(x)$. □

Example 4.7. Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$. Clearly, one may compute $f'(x) = 12x(x^2 - x - 2) = 12x(x + 1)(x - 2)$. This implies that $f' > 0$ for $x \in (-1, 0) \cup (2, \infty)$ and $f' < 0$ for $x \in (-\infty, -1) \cup (0, 2)$. By the increasing/decreasing test, f is increasing on $(-1, 0) \cup (2, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 2)$.

Theorem 4.8 (The first derivative test)**.** *Let f be continuous on I and $c \in I$.*

(1) *If f' changes from positive to negative at c, then f has a local maximum at c.*
(2) *If f' changes from negative to positive at c, then f has a local minimum at c.*
(3) *If f' does not change sign at c, then f has no local extremum at c.*

Proof. For Case (1), assume that there is $\delta > 0$ such that $f'(x) > 0$ for $x \in (c - \delta, c)$ and $f'(x) < 0$ for $x \in (c, c + \delta)$. By the increasing/decreasing test, we have $f(x) < f(y)$ for $c - \delta < x < y < c$ and $f(x) < f(y)$ for $c < y < x < c + \delta$. By the continuity of f at c, this implies that, for $c - \delta < x_1 < c < x_2 < c + \delta$,

$$f(c) = \lim_{y \to c^-} f(y) > f(x_1), \quad f(c) = \lim_{y \to c^+} f(y) > f(x_2).$$

As a result, f has a local maximum at c. The proof of (2) is given by applying (1) at $-f$. For (3), we treat the case $f'(x) > 0$ for $0 < |x - c| < \delta$, while the other is an immediate corollary. As before, the increasing/decreasing test yields $f(x) < f(y)$ for $c - \delta < x < y < c$ and $c < x < y < c + \delta$. By the continuity of f at c, we obtain, for $c - \delta < x_1 < c < x_2 < c + \delta$,

$$f(x_1) < \lim_{y \to c^-} f(y) = f(c) = \lim_{y \to c^+} f(y) < f(x_2).$$

□

Remark 4.6. Note that the differentiability of f at c is not required in the first derivative test.

Example 4.8. For $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, it has been proved before that $f' > 0$ for $x \in (-1, 0) \cup (2, \infty)$ and $f' < 0$ for $x \in (-\infty, -1) \cup (0, 2)$. By the first derivative test, f has a local maximum at 0 and local minima at -1 and 2.

Definition 4.4. Let f be a function defined on an open interval I. f is called

(1) *concave upward* or *convex* if, for any $x \in I$, there is $M_x \in \mathbb{R}$ such that

$$f(y) > f(x) + M_x(y - x), \quad \forall y \in I,\ y \neq x;$$

(2) *concave downward* or *concave* if, for any $x \in I$, there is $M_x \in \mathbb{R}$ such that

$$f(y) < f(x) + M_x(y - x), \quad \forall y \in I,\ y \neq x.$$

Remark 4.7. Let f be a function defined on an open interval I.

(1) f is concave upward if and only if $-f$ is concave downward.
(2) If f is concave upward, then $f(x) < f(x_1) + (x - x_1)[f(x_2) - f(x_1)]/(x_2 - x_1)$ for any $x_1, x_2 \in I$ and any x between x_1 and x_2.

(3) If f is concave upward or downward on I, then f is continuous on I.
(4) Suppose f is concave upward and differentiable at c. Let $M_c \in \mathbb{R}$ be the slope of the supporting line at $(c, f(c))$, i.e. $f(x) > f(c) + (x-c)M_c$ for $x \in I$ and $x \neq c$. Then, one has

$$\frac{f(x)-f(c)}{x-c} > M_c, \quad \forall x > c, \quad \frac{f(x)-f(c)}{x-c} < M_c, \quad \forall x < c.$$

Letting x tend to c implies $M_c = f'(c)$. As a result, if f is differentiable on I, then f is concave upward (resp. downward) if and only if the graph of f lies above (resp. below) or on its tangent lines.

Theorem 4.9 (The concavity test)**.** *Let f be a function defined on an open interval I and assume that f'' exists.*

(1) *If $f''(x) > 0$ for all $x \in I$, then f is concave upward on I.*
(2) *If $f''(x) < 0$ for all $x \in I$, then f is concave downward on I.*

Proof. We prove (1), while (2) is its simple corollary. For $c \in I$, the tangent line to the curve $y = f(x)$ at $(c, f(c))$ is $y = f(c) + (x-c)f'(c)$. To finish the proof, it remains to show that $f(x) > f(c) + (x-c)f'(c)$ for all $x \in I$ and $x \neq c$. Fix $x \in I$ and $x \neq c$. By the mean value theorem, one may select c_x between x and c such that $f(x) - f(c) = f'(c_x)(x-c)$. As f' is differentiable on I, we may choose C_x between c_x and c such that $f'(c_x) - f'(c) = f''(C_x)(c_x - c)$. Immediately, these two identities lead to $f(x) - [f(c) + (x-c)f'(c)] = (x-c)(c_x-c)f''(C_x)$. Note that $c_x < C_x < c$ when $x < c$ and $c_x > C_x > c$ when $x > c$. In addition with the assumption of $f'' > 0$, one has $(x-c)(c_x-c)f''(C_x) > 0$. □

Definition 4.5. A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous at P and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example 4.9. Let $f(x) = x^3 - x$. Note that $f''(x) = 6x$. By the concavity test, f is concave upward on $(0, \infty)$ and downward on $(-\infty, 0)$. This implies that $(0, 0)$ is the inflection point of $y = f(x)$.

Theorem 4.10 (The second derivative test)**.** *Suppose $f'(c) = 0$. If $f'' > 0$ (resp. $f'' < 0$) in a neighborhood of c, then f has a local minimum (resp. local maximum) at c.*

Remark 4.8. Note that $f'(c) = 0$ and $f''(c) \neq 0$ is sufficient to conclude that the local extremum of f at c. Consider the case that $f''(c) > 0$. Set $\epsilon = f''(c)/2$. Since $f''(c)$ exists, there is $\delta > 0$ such that

$$\left| \frac{f'(x) - f'(c)}{x-c} - f''(c) \right| < \epsilon, \quad \forall 0 < |x-c| < \delta.$$

By the triangle inequality, this implies

$$\frac{f'(x)}{x-c} > f''(c) - \epsilon = \frac{f''(c)}{2} > 0, \quad \forall 0 < |x-c| < \delta,$$

which leads to

$$f'(x) \begin{cases} > 0 & \text{for } c < x < c+\delta \\ < 0 & \text{for } c-\delta < x < c \end{cases}.$$

By the increasing/decreasing test, f is decreasing on $(c-\delta, c)$ and increasing on $(c, c+\delta)$. Further, since f is differentiable at c, f is continuous at c. As a consequence, this implies $f(x) > f(c)$ for $0 < |x-c| < \delta$.