4.4. **Indeterminate forms and L'Hôpital's rule.** (Sec. 4.4 in the textbook.)

Definition 4.6. Let f, g be functions defined in a neighborhood of a. The limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ has an indeterminate form of

(1) type $\frac{0}{0}$ if $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$,
(2) type $\frac{\infty}{\infty}$ if $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$.

Theorem 4.11 (L'Hôpital's rule)**.** *Let I be an open interval and $a\in I$. Assume that f, g are differentiable on $I\setminus\{a\}$ with $g'\neq 0$ and the limit of f/g at a has an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. If the limit of f'/g' at a exists or equals ∞ or $-\infty$, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Remark 4.8. L'Hôpital's rule also holds for one sided limits and limits at infinity.

Example 4.10. Let $f(x)=e^x/x^3$. Clearly, the limit of f at ∞ has an indeterminate form of type $\frac{\infty}{\infty}$. By L'Hôpital's rule, one has

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{e^x}{3x^2}=\lim_{x\to\infty}\frac{e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty.$$

Similarly, let $a>0$ and $n\in\mathbb{N}$ be such that $n-1<a\leq n$. By L'Hôpital rule, we obtain

$$\lim_{x\to\infty}\frac{e^x}{x^a}=\lim_{x\to\infty}\frac{e^x}{a\times(a-1)\times\cdots\times(a-n+1)x^{a-n}}=\infty.$$

Example 4.11. Let $a>0$ and $f(x)=x^a/\ln x$. By L'Hôpital's rule, we have

$$\lim_{x\to\infty}\frac{x^a}{\ln x}=\lim_{x\to\infty}\frac{ax^{a-1}}{1/x}=\infty.$$

Remark 4.9. Note that L'Hôpital's rule can fail if a limit is not of indeterminate forms. For instance, applying the rule blindly yields

$$\lim_{x\to\pi^+}\frac{\sin x}{1-\cos x}=\lim_{x\to\pi^+}\frac{(\sin x)'}{(1-\cos x)'}=\lim_{x\to\pi^+}\frac{\cos x}{\sin x}=\infty,$$

which is wrong. In fact, by the limit laws, $\sin x/(1-\cos x)\to 0$ as $x\to\pi^+$.

Example 4.12 (Indeterminate difference $\infty-\infty$). To compute the right limit of $\sec x-\tan x$ at $\pi/2$, we write $\sec x-\tan x=\frac{1}{\cos x}-\frac{\sin x}{\cos x}=\frac{1-\sin x}{\cos x}$. By L'Hôpital's rule, this implies

$$\lim_{x\to(\pi/2)^+}(\sec x-\tan x)=\lim_{x\to(\pi/2)^+}\frac{-\cos x}{-\sin x}=0.$$

Example 4.13 (Indeterminate product $0\cdot\infty$). To see the right limit of $x\ln x$ at 0, we write $x\ln x=\ln x/(1/x)$. This is exactly an indeterminate form of $\frac{\infty}{\infty}$ and, by L'Hôpital's rule,

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=0.$$

Example 4.14 (Indeterminate powers 0^0, ∞^0, 1^∞). Consider the right limit of x^x at 0 and the limit of $(1+cx)^{1/x}$ at 0. For the first one, we write $x^x=\exp\{x\ln x\}$. As exponential functions are continuous, this implies

$$\lim_{x\to 0^+}x^x=\exp\left\{\lim_{x\to 0^+}x\ln x\right\}=1$$

Similarly, the second one is given by

$$\lim_{x\to 0}(1+cx)^{1/x} = \exp\left\{\lim_{x\to 0}\frac{\ln(1+cx)}{x}\right\} = \exp\left\{\lim_{x\to 0}\frac{c}{1+cx}\right\} = e^c.$$

In particular, one has

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x, \quad \forall x\in\mathbb{R}.$$

To prove L'Hôpital's rule, we need Cauchy's mean value theorem.

Lemma 4.12 (Cauchy's mean value theorem)**.** *Let f, g be functions continuous on $[a,b]$ and differentiable on (a,b). Assume that $g'(x)\neq 0$ on (a,b). Then, there is $c\in(a,b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. The proof of Cauchy's mean value theorem is an application of Rolle's theorem to $h(x) = [f(b)-f(a)][g(x)-g(a)] - [g(b)-g(a)][f(x)-f(a)]$, of which details are omitted. □

Proof of L'Hôpital's rule. It suffices to consider the indeterminate forms of one-sided limits. First, assume that $f(x)\to 0$ and $g(x)\to 0$ as $x\to a^+$. Set

$$L = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}, \quad F(x) = \begin{cases} f(x) & \text{if } x>a \\ 0 & \text{if } x=a \end{cases}, \quad G(x) = \begin{cases} g(x) & \text{if } x>a \\ 0 & \text{if } x=a \end{cases}$$

Clearly, F and G are right-continuous at a and the right-limits of f/g and F/G at a coincide.

Let $\delta>0$ be such that $(a,a+\delta)\subset I$. Note that, for $x\in(a,a+\delta)$, F and G are continuous on $[a,x]$ and differentiable on (a,x) with $G'(x)\neq 0$. By Cauchy's mean value theorem, there is $y\in(a,x)$ such that

$$\left(\frac{f'(y)}{g'(y)}=\right)\frac{F'(y)}{G'(y)} = \frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F(x)}{G(x)}\left(=\frac{f(x)}{g(x)}\right).$$

As $x\to a^+$ implies $y\to a^+$, the above identities leads to $f(x)/g(x)\to L$ as $x\to a^+$.

Next, assume that $f(x)\to\infty$, $g(x)\to\infty$ and $f'(x)/g'(x)\to L$ as $x\to a^+$. We treat the case of $L\in\mathbb{R}$, while the other case can be proved similarly and omitted. Let $\epsilon>0$. Since $f'(x)/g'(x)\to L$ as $x\to a^+$, we may choose $\delta_1>0$ such that $|f'(x)/g'(x)-L|<\epsilon/6$ for $a<x<a+\delta_1$. Set $x_0=a+\delta_1/2$ and let $x\in(a,x_0)$. Note that f,g are continuous on $[x,x_0]$, differentiable on (x,x_0) and $g'\neq 0$ on (x,x_0). By Cauchy's mean value theorem, there is $y\in(x,x_0)$ such that

$$\frac{f'(y)}{g'(y)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)/g(x)-f(x_0)/g(x)}{1-g(x_0)/g(x)}.$$

As $y\in(x,x_0)\subset(a,a+\delta_1)$, we have, for $a<x<x_0$,

$$\left|\frac{f(x)/g(x)-L}{1-g(x_0)/g(x)}\right| - \left|\frac{g(x_0)L-f(x_0)}{g(x)-g(x_0)}\right| \le \left|\frac{f(x)/g(x)-f(x_0)/g(x)}{1-g(x_0)/g(x)} - L\right| < \frac{\epsilon}{6}.$$

Since $g(x)\to\infty$ as $x\to a^+$, one may select $0<\delta_2<\delta_1/2$ such that

$$\left|\frac{g(x_0)L-f(x_0)}{g(x)-g(x_0)}\right| < \frac{\epsilon}{3}, \quad 0 < \left|1-\frac{g(x_0)}{g(x)}\right| < 2, \quad \forall a<x<a+\delta_2.$$

Consequently, we obtain $|f(x)/g(x)-L|<\epsilon$ for $a<x<a+\delta_2$. □