4.5. **Summary of curve sketching.** (Sec. 4.5 in the textbook)

The following are guidelines for graphing.

(1) Determine the domain, intercepts and symmetry (even, odd or periodic).
(2) Find vertical and horizontal asymptotes.
(3) Determine intervals of increasing and decreasing by the first derivative.
(4) Determine the convexity and inflection points by the second derivative.
(5) Check local extrema and their values by the first or second derivative tests.

*Example* 4.15. Let $f(x) = \frac{2x^2}{x^2-1}$. The domain of $f$ is $D = \{x \in \mathbb{R} : |x| \neq 1\}$ and $(0,0)$ is the only intercept. Clearly, $f$ is an even function and

$$\lim_{x\to\infty} f(x) = 2, \quad \lim_{x\to 1^+} f(x) = \infty, \quad \lim_{x\to 1^-} f(x) = -\infty.$$

In some basic computations, one has $f'(x) = \frac{-4x}{(x^2-1)^2}$ and $f''(x) = \frac{4(3x^2+1)}{(x^2-1)^3}$. This implies

$$f' < 0 \text{ on } (0,\infty), \quad f'' > 0 \text{ on } (1,\infty), \quad f'' < 0 \text{ on } (0,1).$$

Note that $f$ has a local maximum at 0, no local minimum and inflection point.

*Example* 4.16. Let $g(x) = xe^x$. Clearly, the origin is the only intercept and

$$\lim_{x\to\infty} g(x) = \infty, \quad \lim_{x\to-\infty} g(x) = 0.$$

Note that $g'(x) = e^x(x+1)$ and $g''(x) = e^x(x+2)$. Immediately, this implies

$$g' < 0 \text{ on } (-\infty,-1), \quad g' > 0 \text{ on } (-1,\infty), \quad g'' < 0 \text{ on } (-\infty,-2), \quad g'' > 0 \text{ on } (-2,\infty).$$

One can see that $g$ has a local minimum at $-1$, no local maximum but has an inflection point at $(2, g(2))$.

*Example* 4.17. Let $h(x) = \frac{x^2}{x+1}$. The domain of $h$ is $(-\infty,-1) \cup (-1,\infty)$, the origin is the unique intercept and

$$\lim_{x\to(-1)^+} h(x) = \lim_{x\to\infty} h(x) = \infty, \quad \lim_{x\to(-1)^-} h(x) = \lim_{x\to-\infty} h(x) = -\infty.$$

Note that $h'(x) = 1 - \frac{1}{(x+1)^2}$ and $h''(x) = \frac{2}{(x+1)^3}$. This implies

$$h' < 0 \text{ on } (-2,0), \quad h' > 0 \text{ on } (-\infty,-2) \cup (0,\infty)$$

and

$$h'' < 0 \text{ on } (-\infty,-1), \quad h'' > 0 \text{ on } (-1,\infty).$$

Hence, $h$ has a local maximum at $-2$, a local minimum at 0 but no inflection point.

**Definition 4.7.** A graph $y = f(x)$ has $y = mx + b$ as a *slant asymptote* as $x \to \infty$ (resp. $x \to -\infty$) if

$$\lim_{x\to\infty} [f(x) - mx + b] = 0 \quad (\text{resp. } \lim_{x\to-\infty} [f(x) - mx + b] = 0).$$

*Remark* 4.10. In the previous example, $h$ has slant asymptote $y = x - 1$ as $x \to \pm\infty$.

*Remark* 4.11. Mostly, slant asymptotes appear in rational functions of which degree in the numerator is exactly larger than that of the denominator by one.