4.6. **Optimization problems.** (Sec. 4.7 in the textbook)

Example 4.18. Consider the areas of rectangles inscribed in the circle $x^2 + y^2 = 1$. Let (x, y) be the point of the intercept of a rectangle and the unit circle in the first quadrant. Then, the area of this rectangle is $A = 4xy$. To see the maximum of A, we write $x = \cos\theta$ and $y = \sin\theta$ with $\theta \in (0, \pi/2)$. Clearly, A is a function of θ given by $A(\theta) = 4\sin\theta\cos\theta = 2\sin(2\theta)$. Note that

$$A'(\theta) = 4\cos(2\theta) \begin{cases} > 0 & \text{for } \theta \in (0, \pi/4) \\ < 0 & \text{for } \theta \in (\pi/4, \pi/2) \end{cases}.$$

By the increasine/decreasin test, A is increasing on $(0, \pi/4)$ and decreasing on $(\pi/4, \pi/2)$. This implies that $\theta = \pi/4$ is the maximum of A on $(0, \pi/2)$ with value $A(\pi/4) = 2$.

Example 4.19. The R&D department in a can factory plans to design an one-liter cylindrical can and want to know what dimensions minimize the usage of tin. Let h and r be the height and the radius of the bottom. Then, the dimensions satisfy $\pi r^2 h = 1000$ and the surface area of the can is $A(r) = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{2000}{r}$ with $r \in (0, \infty)$. Note that

$$A'(r) = 4\pi r - \frac{2000}{r^2}, \quad A''(r) = 4\pi + \frac{4000}{r^3}.$$

This implies that $A'(r_0) = 0$, where $r_0 = (500/\pi)^{1/3}$, and $A'' > 0$. By the increasing/decreasing test, $A' < 0$ on $(0, r_0)$ and $A' > 0$ on (r_0, ∞). Consequently, A is decreasing on $(0, r_0)$ and increasing on (r_0, ∞). Hence, r_0 is the global minimum of A on $(0, \infty)$.

Example 4.20. A cellular phone is sold at price \$350 and the weekly sale volume is 200 units. The customer census shows that the weekly sale volume evolves linearly and increase 20 units if a rebate of \$10 is made. To see the price that maximizes the revenue, let x be the weekly sale volume, $p(x)$ be the corresponding price of a unit and $R(x)$ be the weekly revenue. Immediately, one has

$$p(x) = 350 - \frac{10}{20}(x - 200) = 450 - \frac{x}{2}, \quad R(x) = xp(x) = 450x - \frac{x^2}{2}.$$

Note that $R'(x) = 450 - x$. By the first derivative test, R is increasing on $(0, 450)$ and decreasing on $(450, \infty)$. This implies that R has its global maximum at $x = 450$ and the corresponding price is $p(450) = 225$.