4.7. **Antiderivatives.** (Sec. 4.9 in the textbook)

Definition 4.8. F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Theorem 4.13. *If F and G are both antiderivatives of f on an interval I, then there is $c \in \mathbb{R}$ such that $G(x) = F(x) + c$ for all $x \in I$.*

Remark 4.12. In general, when F is an antiderivative of f on an interval I, we shall write $F(x) + C$, where C is a constant, for any antiderivative of f on I.

Remark 4.13. If F, G are antiderivatives of f, g, then cF and $F + G$ are antiderivatives of cf and $f + g$.

Example 4.21. Let $f_1(x) = \cos x$, $f_2(x) = 1/x$, $f_3(x) = x^r$ with $r \neq -1$ and $f_4(x) = e^x$. Then, their antiderivatives are $F_i + C$, where

$$F_1(x) = \sin x, \quad F_2(x) = \ln x, \quad F_3(x) = \frac{x^{r+1}}{r+1}, \quad F_4(x) = e^x.$$

Example 4.22. Consider the antiderivative F of $f(x) = e^x + \frac{20}{1+x^2}$ satisfying $F(0) = -2$. Note that e^x and $\tan^{-1} x$ are antiderivatives of e^x and $\frac{1}{1+x^2}$. This implies $F(x) = e^x + 20\tan^{-1} x + C$ for some $C \in \mathbb{R}$. Letting $x = 0$ implies $C = F(0) - e^0 - 20\tan^{-1} 0 = -3$. Hence, $F(x) = e^x + 20\tan^{-1} x - 3$.

Example 4.23. A car moves in a straight line with acceleration $a(t) = 6t + 4$. Assume that the velocity at time 0 is $v(0) = -6$ and the position at time 1 is $s(1) = 2$. Note that $s' = v$ and $v' = a$. By the second identity, one has $v(t) = 3t^2 + 4t + C_1$. Letting $t = 0$ implies $C_1 = v(0) = -6$, which yields $v(t) = 3t^2 + 4t - 6$. Similarly, we obtain $s(t) = t^3 + 2t^2 - 6t + C_2$. By applying $t = 1$ to the equation, we obtain $C_2 = s(1) + 3 = 5$ and, hence, $s(t) = t^3 + 2t^2 - 6t + 5$.

Example 4.24. Consider a ball thrown upward at the speed of 15m/s from the edge of a cliff 140m above the ground. To record the trajectory of the ball, let $a(t)$, $v(t)$ and $s(t)$ be the functions of acceleration, velocity and position. Then, one has $s'(t) = v(t)$, $v'(t) = a(t)$ and $a(t) = -9.8$ for $t \geq 0$. This implies $v(t) = -9.8t + C_1$ and then $s(t) = -4.9t^2 + C_1 t + C_2$. Along with the initial conditions, $s(0) = 140$ and $v(0) = 15$, we obtain $C_1 = v(0) = 15$ and $C_2 = 140$. Thus, $s(t) = -4.9t^2 + 15t + 140$. By writing

$$s(t) = -4.9\left(t - \frac{15}{9.8}\right)^2 + 140 - \frac{225}{19.6},$$

it is easy to see that the ball reaches the maximum height at time $15/9.8$ with height $225/19.6 \approx 151.48$m.