5.1. **Areas and distances.** (Sec. 5.1 in the textbook)

Let $f(x) = x^2$ and $S = \{(x,y)|0 \le x \le 1, 0 \le y \le f(x)\}$. To see the area A of S, let's partition $[0,1]$ into $[0,1/n]$, $[1/n,2/n]$, ..., $[1-1/n,1]$ and set

$$R_n = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \times \frac{1}{n}, \quad L_n = \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right) \times \frac{1}{n}.$$

Since f is increasing on $[0,1]$, one has $L_n < A < R_n$. Note that $L_n = R_n - 1/n$ and

$$R_n = \frac{(n+1)(2n+1)}{6n^2} \to \frac{1}{3}, \quad \text{as } n \to \infty.$$

By the squeeze theorem, we obtain $A = 1/3$.

In the same spirit, for any continuous function f on $[a,b]$, we set $\Delta x = (b-a)/n$, $x_i = a+i\Delta x$ and

$$L_n = \Delta x \sum_{i=0}^{n-1} f(x_i), \quad R_n = \Delta x \sum_{i=1}^{n} f(x_i).$$

By the (uniform) continuity of f on $[a,b]$, $L_n - R_n \to 0$ as $n \to \infty$.

Definition 5.1. Let f be a nonnegative continuous function on $[a,b]$. Then, the area of the region $\{(x,y)|a \le x \le b, 0 \le y \le f(x)\}$ is defined to be the limit of R_n.

In the above definition, if the endpoints are replaced with any point in $[x_{i-1},x_i]$, say x_i^*, then one obtains a new sequence to approximate the area. In this case, we call $x_1^*,...,x_n^*$ *sample points*. Particularly, if f has its maximum and minimum over $[x_{i-1},x_i]$ at x_i^* and y_i^* (by the extremum value theorem), then the continuity of f on $[a,b]$ implies $\Delta x \sum_{i=1}^{n}[f(x_i^*)-f(y_i^*)] \to 0$ as $n \to \infty$. This implies that, for any sample point x_i^*, $\Delta x \sum_{i=1}^{n} f(x_i^*)$ has the same limit, which equals to the limit of R_n and L_n.

Example 5.1. Consider a car of which odometer is broken. To estimate the driven distance over a period of 30 seconds, the driver reads the speedometer every 5 seconds and the record is as follows.

Time (seconds)	0	5	10	15	20	25	30
Velocity (km/h)	27	34	38	46	51	50	45

Through the equality 1 km/h=1/3.6 m/s, the above table turns out the following one.

Time (seconds)	0	5	10	15	20	25	30
Velocity (m/s)	7.5	9.4	10.6	12.8	14.2	13.9	12.5

Using the velocity at the beginning of each 5-second period as the average velocity, the travelling distance is given by

$$7.5 \cdot 5 + 9.4 \cdot 5 + 10.6 \cdot 5 + 12.8 \cdot 5 + 14.2 \cdot 5 + 13.9 \cdot 5 = 342.$$

Similarly, if the velocity at the end of each 5-second period is regarded as the average velocity, then the travelling distance becomes

$$9.4 \cdot 5 + 10.6 \cdot 5 + 12.8 \cdot 5 + 14.2 \cdot 5 + 13.9 \cdot 5 + 12.5 \cdot 5 = 367.$$