5.2. **The definite integral.** (Sec. 5.2 in the textbook)

Definition 5.2. Let f be a function defined on $[a,b]$. Set $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ and let $x_i^* \in [x_{i-1}, x_i]$ for $1 \le i \le n$. The *definite integral* of f from a to b is denoted and defined by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x, \tag{5.1}$$

provided the limit exists and is independent of the choice of sample points. In this case, we say that f is *integrable* on $[a,b]$.

Remark 5.1. The symbol of $\int$ is introduced by Leibniz and called the *integral sign*. In the notation $\int_a^b f(x)dx$, f is called the *integrand* and a, b are called the *limits of integration*, whereas a refers to the *lower limit* and b refers to the *upper limit*. The procedure of calculating an integral is called *integration*.

Remark 5.2. The summation in the right hand side of (5.1) is called the *Riemann sum*.

Remark 5.3. Among all choices of sample points, the Riemann sum with midpoints,

$$\Delta x \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right),$$

is generally closer to the desired integral than those with left or right endpoints.

Remark 5.4. In fact, one may consider non-uniform partitions of $[a,b]$ in Definition 5.2. In detail, let $x_0 = a < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$, set $\Delta x_i = x_i - x_{i-1}$ and select $x_i^* \in [x_{i-1}, x_i]$. More rigorously, we say that f is integrable on $[a,b]$ if the following limit

$$\lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x_i \tag{5.2}$$

exists and independent of the choice of $x_0, ..., x_n$ satisfying $\max\{\Delta x_i | 1 \le i \le n\} \to 0$ and independent of the selection of sample points. Actually, the definition in (5.2) is consistent with the definition in (5.1).

Example 5.2. Let f, g be functions defined by

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q} \end{cases}, \quad g(x) = \begin{cases} 1 & \text{if } x \ne 1/2 \\ 0 & \text{if } x = 1/2 \end{cases}.$$

Then, f is not integrable but g is integrable.

Theorem 5.1. *Let $\Delta x, x_i$ be as in Definition 5.2. If f is integrable on $[a,b]$, then*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x. \tag{5.3}$$

Theorem 5.2. *If f is continuous on $[a,b]$ or has only finitely many jump discontinuities, then f is integrable on $[a,b]$.*

Example 5.3. Let $f(x) = x^3 + x\sin x$. Since f is continuous on $[0,\pi]$,

$$\int_0^\pi f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n (x_i^3 + x_i \sin x_i)\Delta x, \quad x_i = \frac{i\pi}{n}, \quad \Delta x = \frac{\pi}{n}.$$

Some formulas on the evaluation of integrals

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

and

$$\sum_{i=1}^n c = nc, \quad \sum_{i=1}^n ca_i = c\sum_{i=1}^n a_i, \quad \sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i.$$

Example 5.4. Let $f(x) = x^3 - 6x$ and set $R_n = \sum_{i=1}^n f(x_i)\Delta x$, where $\Delta x = 3/n$ and $x_i = 3i/n$. Note that

$$R_n = \frac{3}{n}\sum_{i=1}^n \left[\left(\frac{3i}{n}\right)^3 - 6\cdot\frac{3i}{n}\right] = \frac{3}{n}\left[\frac{27}{n^3}\sum_{i=1}^n i^3 - \frac{18}{n}\sum_{i=1}^n i\right] = \frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right).$$

By Theorem 5.2, $\int_0^3 (x^3 - 6x)dx = -27/4$.

Example 5.5. To compute $\int_1^2 e^x dx$, we need to estimate its Riemann sum. Note that

$$R_n = \frac{1}{n}\sum_{i=1}^n e^{1+i/n} = \frac{e}{n}\sum_{i=1}^n \left(e^{1/n}\right)^i = \frac{e}{n}\cdot\frac{e^{1/n}[e^{n/n}-1]}{e^{1/n}-1} = (e^2 - e)\frac{1/n}{1-e^{-1/n}}.$$

By L'Hôpital's rule, this implies

$$\int_1^2 e^x dx = \lim_{n\to\infty} R_n = (e^2 - e)\lim_{x\to 0^+}\frac{x}{1-e^{-x}} = (e^2 - e)\lim_{x\to 0^+}\frac{1}{e^{-x}} = e^2 - e.$$

Example 5.6. To evaluate the integrals of $\int_0^1 \sqrt{1-x^2}dx$ and $\int_0^1 (x-1)dx$, one may interpret them as the areas of some regions and obtain the results of $\pi/4$ and $-1/2$.

Properties of definite integrals Let f, g be integrable functions and $a, b, c, A, m, M \in \mathbb{R}$.

$$(1) \int_b^a f(x)dx = -\int_a^b f(x)dx, \quad \forall a < b; \quad (2) \int_a^a f(x)dx = 0;$$

$$(3) \int_a^b Adx = A(b-a); \quad (4) \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx;$$

$$(5) \int_a^b Af(x)dx = A\int_a^b f(x)dx; \quad (6) \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx;$$

$$(7)\, f(x) \geq g(x),\ \forall a \leq x \leq b, \quad \Rightarrow \quad \int_a^b f(x)dx \geq \int_a^b g(x)dx;$$

$$(8)\, m \leq f(x) \leq M,\ \forall a \leq x \leq b, \quad \Rightarrow \quad m(b-a) \leq \int_a^b f(x)dx \leq \int M(b-a).$$

The above properties follows immediately from the definition of integration and the limit laws, whereas the details are omitted.