5.3. **The fundamental theorem of calculus.** (Sec. 5.3 in the textbook)

**Theorem 5.3** (The first part of the fundamental theorem of calculus)**.** *Let $f$ be continuous on $[a,b]$ and set*

$$g(x) = \int_a^x f(t)dt \quad \forall x \in [a,b].$$

*Then, $g$ is continuous on $[a,b]$, differentiable on $(a,b)$ and $g'(x) = f(x)$ for $x \in (a,b)$.*

*Proof.* Let $x \in (a,b)$ and $h > 0$ such that $x + h \in [a,b]$. Write

$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt.$$

Since $f$ is continuous on $[x, x+h]$, we may choose (by the extremum value theorem) $u, v \in [x, x+h]$ such that $f(u) \le f(t) \le f(v)$ for $t \in [x, x+h]$. This implies

$$f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v).$$

Note that $u, v \to x$, when $h \to 0$. By the continuity of $f$ and the Squeeze theorem, we obtain

$$\lim_{h\to 0^+} \frac{g(x+h) - g(x)}{h} = f(x), \quad \forall x \in [a,b), \quad \lim_{h\to 0^+} g(a+h) = g(a).$$

A similar statement as before yields

$$\lim_{h\to 0^-} \frac{g(x+h) - g(x)}{h} = f(x), \quad \forall x \in (a,b], \quad \lim_{h\to 0^-} g(b+h) = g(b).$$

As a consequence, $g'(x) = f(x)$ for $x \in (a,b)$ and $g$ is continuous on $[a,b]$. □

*Remark* 5.5. If $f$ is continuous on $[a,b]$, then

$$\frac{d}{dx} \int_a^x f(t)dt = \frac{d}{dx} \int_b^x f(t)dt = f(x), \quad \forall a < x < b.$$

*Example* 5.7. To differentiate $\int_0^x \sin t^2 dt$ and $\int_0^{x^2} \sin t^2 dt$, we set $f(t) = \sin t^2$ and $g(x) = \int_0^x f(t)dt$. By the fundamental theorem of calculus, this implies

$$g'(x) = \frac{d}{dx} \int_0^x \sin t^2 dt = f(x) = \sin x^2$$

and

$$\frac{d}{dx} \int_0^{x^2} \sin t^2 dt = \frac{d}{dx} g(x^2) = g'(x^2) \times 2x = 2x \sin x^4.$$

**Theorem 5.4** (The second part of the fundamental theorem of calculus)**.** *Suppose $f$ is continuous on $[a,b]$ and let $F$ be an antiderivative of $f$ on $(a,b)$, which is continuous on $[a,b]$. Then,*

$$\int_a^b f(t)dt = F(x)\Big|_a^b := F(b) - F(a)$$

*Proof.* Set $g(x) = \int_a^x f(t)dt$. By the first part of the fundamental theorem of calculus, $g' = f$ on $(a,b)$. Since $F' = f$ on $(a,b)$, there is $C \in \mathbb{R}$ such that $g = F + C$ on $(a,b)$. As $g$ and $F$ are continuous on $[a,b]$, one has $g = F + C$ on $[a,b]$. Consequently, this implies

$$0 = \int_a^a f(t)dt = g(a) = F(a) + C, \quad \int_a^b f(t)dt = g(b) = F(b) + C.$$

□

*Example* 5.8. Consider the integrals of $\int_1^2 e^x dx$, $\int_a^b \frac{1}{\sqrt{1-x^2}} dx$ and $\int_1^2 \frac{dx}{x}$. Note that the antiderivatives of $e^x$, $\frac{1}{\sqrt{1-x^2}}$ and $\frac{1}{x}$ are $e^x$, $\sin^{-1} x$ and $\ln x$. By the second part of the fundamental theorem of calculus, we have $\int_1^2 e^x dx = e^2 - e$, $\int_a^b \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} b - \sin^{-1} a$ for $-1 < a < b < 1$ and $\int_1^2 \frac{dx}{x} = \ln 2 - \ln 1 = \ln 2$.

Both parts of the fundamental theorem of calculus can be summarized as follows.

**Theorem 5.5** (The fundamental theorem of calculus). *Let $f$ be a continuous function on $[a, b]$ and $F$ be an antiderivative of $f$ on $(a, b)$ which is continuous on $[a, b]$. Then,*

$$\frac{d}{dx} \int_a^x f(t)dt = f(x), \quad \forall x \in (a, b), \qquad \int_a^b f(t)dt = F(x)\Big|_a^b = F(b) - F(a).$$

*Remark* 5.6. It is remarkable that if $f$ is continuous on $[a, b]$, then there is always an antiderivative of $f$ on $(a, b)$ that is continuous on $[a, b]$.