5.4. **Indefinite integrals and the net change theorem.** (Sec. 5.4 in the textbook)

Definition 5.3. For an integrable function f, its indefinite integral is denoted by $\int f(x)dx$ and refers to an antiderivatives of f.

Remark 5.7. Note that a definite integral is a number but an indefinite integral is a function or a family of functions.

Example 5.9. For the function $f(x) = x$, $\int f(x)dx = \frac{x^2}{2} + C$, where C is a constant.

Table of some indefinite integrals

$$\int cf(x)dx = c\int f(x)dx; \quad \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx;$$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C, \quad \forall r \neq -1; \quad \int \frac{1}{x}dx = \ln|x| + C; \quad \int b^x dx = \frac{b^x}{\ln b} + C$$

$$\int \cos x dx = \sin x + C; \quad \int \sec^2 x dx = \tan x + C; \quad \int \sec x \tan x dx = \sec x + C;$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C; \quad \int \frac{1}{1+x^2}dx = \tan^{-1} x + C.$$

Example 5.10. By writing $\sin x(\cos x)^{-2} = \tan x \sec x$, one has $\int \sin x(\cos x)^{-2}dx = \sec x + C$.

Example 5.11. To determine $\int_0^3 (x^3 - 6x)dx$, we first compute $\int (x^3 - 6x)dx = \frac{x^4}{4} - 3x^2 + C$. By the fundamental theorem of calculus, this implies

$$\int_0^3 (x^3 - 6x)dx = \left(\frac{x^4}{4} - 3x^2 + C\right)\Big|_0^3 = \frac{-27}{4}.$$

Remark 5.8. Since the function x^{-2} has a separated domain $(-\infty, 0) \cup (0, \infty)$, the indefinite integral of x^{-2} can be expressed in another form, say

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & \text{on } (-\infty, 0) \\ -\frac{1}{x} + C_2 & \text{on } (0, \infty) \end{cases}.$$

Example 5.12. Consider a car moving on a straight line with velocity function $v(t) = t^2 - t - 6$ and position function $s(t)$. As $s' = v$, one may choose $C \in \mathbb{R}$ such that $s(t) = t^3/3 - t^2/2 - 6t + C$. This implies that the displacement of the car during period $[0, 4]$ is

$$s(4) - s(0) = \frac{t^3}{3} - \frac{t^2}{2} - 6t\Big|_0^4 = -\frac{32}{3}.$$

Since $v < 0$ on $(-2, 3)$ and $v > 0$ on $(-\infty, 2) \cup (3, \infty)$, the travelling distance during period $[0, 4]$ is

$$\int_0^4 |v(t)|dt = -\int_0^3 v(t)dt + \int_3^4 v(t)dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t\Big|_3^0 + \frac{t^3}{3} - \frac{t^2}{2} - 6t\Big|_3^4 = \frac{49}{3}.$$