5.5. **The substitution rules.** (Sec. 5.5 in the textbook)

Let's start with the indefinite integral $\int 2x\sqrt{x^2+1}dx$. It would not be easy to see an antiderivative of $2x\sqrt{x^2+1}$ unless specific techniques are introduced. By writing $u = x^2+1$, one has $du = 2xdx$. A formal replacement of the above notation leads to

$$\int 2x\sqrt{x^2+1}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2+1)^{3/2} + C.$$

Theorem 5.6. *If $u = g(x)$ is differentiable and f is continuous, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

In particular, if F is an antiderivative of f, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Proof. Set $v = \int f(u)du$. By the chain rule, $\frac{dv}{dx} = \frac{dv}{du}\cdot\frac{du}{dx} = f(u)g'(x) = f(g(x))g'(x)$. □

Example 5.13. To determine $\int \frac{1}{\sqrt{x+1}}dx$, we set $u = x+1$. Since $du = dx$ and $\int u^{-1/2}du = 2u^{1/2}$, Theorem 5.6 implies $\int \frac{1}{\sqrt{x+1}}dx = 2\sqrt{x+1} + C$.

Example 5.14. By Theorem 5.6, if $f(u) = 1/u$ and g is differentiable, then $\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + C$. In particular, $\int \tan x dx = -\ln|v| + C = -\ln|\cos x| + C = \ln|\sec x| + C$.

Theorem 5.7. *If g' is continuous on $[a,b]$ and f is continuous on $g([a,b])$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. The continuity of f and g' guarantees the integrability of $f(g(x))g'(x)$ over $[a,b]$. By the extremum value theorem, the maximum and minimum values, say M and m, of g on $[a,b]$ exist and belong to $g([a,b])$. By the first part of the fundamental theorem of calculus, f has an antiderivative, say F, on (m,M), which is continuous on $[m,M]$. By Theorem 5.6, $F(g(x))$ equals $\int f(g(x))g'(x)dx$ on (a,b) and is continuous on $[a,b]$. By the second part of the fundamental theorem of calculus, we obtain

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du.$$

□

Example 5.15. To estimate $\int_1^e \frac{\ln x}{x}dx$, let $u = \ln x$ and $f(u) = u$. By Theorem 5.7, this implies $\int_1^e \frac{\ln x}{x}dx = \int_0^1 udu = \frac{u^2}{2}\big|_0^1 = \frac{1}{2}$.

Theorem 5.8 (Integrals of symmetric functions)**.** *Suppose that f is continuous on $[-a,a]$. If f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$. If f is odd, then $\int_{-a}^a f(x)dx = 0$.*

Example 5.16. Since $(x^2+1)\sin x$ is odd, $\int_{-\pi}^{\pi}(x^2+1)\sin x dx = 0$. In fact, the indefinite integral of $(x^2+1)\sin x$ can be complicated.