6. Applications of integrations

6.1. **Areas between curves.** (Sec. 6.1 in the textbook)

Theorem 6.1. *Let f, g be continuous functions on $[a,b]$ and A be the area of the region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$. Then,*

$$A = \int_a^b |f(x) - g(x)| dx.$$

In particular, if $f \geq g$ on $[a,b]$, then $A = \int_a^b [f(x) - g(x)] dx$.

Example 6.1. Find the area A of the region bound by $y = \sqrt{x}$, $y = x/2$, $x = 2$ and $x = 8$. Note that $\sqrt{x} > x/2$ if and only if $0 < x < 4$. This implies

$$\begin{aligned} A &= \int_2^8 |\sqrt{x} - x/2| dx = \int_2^4 (\sqrt{x} - x/2) dx + \int_4^8 (x/2 - \sqrt{x}) dx \\ &= \left(\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right)\Big|_2^4 + \left(\frac{x^2}{4} - \frac{2}{3}x^{3/2}\right)\Big|_4^8 = \frac{59}{3} - 12\sqrt{2} \end{aligned}$$

Example 6.2. Find the area A of the region enclosed by $y = \sin x$ and $y = \cos x$, $x = 0$ and $x = \pi/2$. Note that $\sin x < \cos x$ on $[0, \pi/4)$ and $\sin x > \cos x$ on $(\pi/4, \pi/2]$. Then, one has

$$\begin{aligned} A &= \int_0^{\pi/2} |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\ &= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\sin x - \cos x)\Big|_{\pi/4}^{\pi/2} = 2(\sqrt{2} - 1). \end{aligned}$$

Example 6.3. Find the area A of the region S enclosed by $y = x - 1$ and $y^2 = 2x + 6$. Note that $x = y + 1 = \frac{y^2}{2} - 3$ if and only if $y \in \{-2, 4\}$.
Method 1: We split S into two regions S_1, S_2 using line $x = -1$, where $S_1 = \{(x,y) | x \in [-3,-1], |y| \leq \sqrt{2x+6}\}$ and $S_2 = \{(x,y) | x \in [-1,5], x - 1 \leq y \leq \sqrt{2x+6}\}$. Then,

$$\begin{aligned} A &= \int_{-3}^{-1} [\sqrt{2x+6} - (-\sqrt{2x+6})] dx + \int_{-1}^{5} [\sqrt{2x+6} - (x-1)] dx \\ &= \frac{2}{3}(2x+6)^{3/2}\Big|_{-3}^{-1} + \left(\frac{1}{3}(2x+6)^{3/2} - \frac{x^2}{2} + x\right)\Big|_{-1}^{5} = 18. \end{aligned}$$

Method 2: By writing $S = \{(x,y) | y \in [-2,4], y^2/2 - 3 \leq x \leq y + 1\}$, we have

$$A = \int_{-2}^{4} \left[y + 1 - \left(\frac{y^2}{2} - 3\right)\right] dy = \left(-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right)\Big|_{-2}^{4} = 18.$$