6.2. **Volumes.** (Sec. 6.2 in the textbook)

Consider a solid $S \subset \mathbb{R}^3$ and let V be its volume. To compute V, we use the plane $x = a$ to cut S and obtain a plane region $S(a) = \{(x, y, z) \in S | x = a\}$. $S(a)$ is called a *cross-section* of S and we use $A(a)$ to denote its area. Following the idea of approximating rectangles in the Riemann sum, let's partition $[a, b]$ into n subintervals of width Δx and choose $x_i^* \in [x_{i-1}, x_i]$ as a sample point as before. By using blocks $[x_{i-1}, x_i] \times S(x_i^*)$ as approximating cylinders, we obtain a Riemann sum for the volume of S as follows,

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

Definition 6.1. Let S be a solid lying between $x = a$ and $x = b$. Suppose the cross-sectional area $A(x)$ is continuous on $[a, b]$. The volume V of S is defined by

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx,$$

provided the limit exists.

Example 6.4. Let S be the ball $x^2 + y^2 + z^2 \le r^2$. Note that the cross section of S at x is a disc of radius $\sqrt{r^2 - x^2}$. This implies $A(x) = (r^2 - x^2)\pi$ and

$$V = \int_{-r}^{r} A(x)dx = \pi(r^2x - x^3/3)\Big|_{-r}^{r} = \frac{4\pi}{3}r^3.$$

Example 6.5. Let S be the solid obtained by rotating about the x-axis the region $\{(x, y)|0 \le y \le \sqrt{x}, 0 \le x \in 1\}$. The cross section at x is a disc of radius $\sqrt{x}$. This implies $A(x) = x\pi$ and then

$$V = \int_0^1 A(x)dx = \frac{x^2\pi}{2}\Big|_0^1 = \frac{\pi}{2}.$$

Example 6.6. Let R be the region enclosed by $y = x^2$ and $y = x^3$ and S be the solid obtained by rotating R about the x-axis. Note that $R = \{(x, y)|0 \le x \le 1, x^3 \le y \le x^2\}$. The cross section $S(x)$ is an annulus of which inside and outside circles are of radii x^3 and x^2. This implies $A(x) = \pi(x^4 - x^6)$ and

$$V = \int_0^1 A(x)dx = \pi\left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Big|_0^1 = \frac{2\pi}{35}.$$

Example 6.7. Consider a pyramid of which base is a square with width L and of which height is h. To compute the volume of the pyramid, we place the peak of the pyramid at the origin and overlap the axis of the pyramid and the x-axis. The cross section of the pyramid at x is a regular rectangle with width $\frac{Lx}{h}$. This implies $A(x) = L^2x^2/h^2$ and

$$V = \int_0^h A(x)dx = \frac{L^2x^3}{3h^2}\Big|_0^h = \frac{L^2h}{3}.$$