6.3. **Volumes by cylindrical shells.** (Sec. 6.3 in the textbook)

To compute the volume of the solid obtained by rotating the region $\{(x,y)|a \le x \le b,\, y \text{ between } f(x) \text{ and } 0\}$ about the y-axis, we may use the *method of cylindrical shells*. Suppose that we have a cylindrical shell of which inner and outer radii are r_1 and r_2 with height h. Then, its volume is

$$V = \pi(r_2^2 - r_1^2)h = 2\pi\frac{r_1 + r_2}{2}(r_2 - r_1)h.$$

By setting $\Delta r = r_2 - r_1$ and $r = (r_1 + r_2)/2$, the volume can be rewritten as $V = 2\pi rh\Delta r$.

Let S be the solid obtained by rotating the region enclosed by $y = f(x)$, $x = a \ge 0$, $x = b$ and $y = 0$ about the y-axis. By partitioning $[a, b]$ into n subintervals of equal width, we may approximate the volume V of S using those of the cylindrical shells and obtain

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\overline{x}_i|f(\overline{x}_i)|\Delta x,$$

where $\overline{x}_i = (x_{i-1} + x_i)/2$.

Theorem 6.2. *Let f be a continuous function on $[a, b]$ with $a \ge 0$. The volume of the solid obtained by rotating the region bounded by $y = f(x)$, $x = a$, $x = b$ and $y = 0$ about the y-axis is equal to $2\pi\int_a^b x|f(x)|dx$.*

Example 6.8. Let S be the solid obtained by rotating the region enclosed by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. Note that $2x^2 - x^3 = 0$ if $x \in \{0, 2\}$. Then, the volume of S equals

$$2\pi\int_0^2 x|2x^2 - x^3|dx = 2\pi\int_0^2 (2x^3 - x^4)dx = 2\pi\left(\frac{x^4}{2} - \frac{x^5}{5}\right)\Bigg|_0^2 = \frac{16\pi}{5}.$$

Theorem 6.3. *Let f, g be continuous functions on $[a, b]$ with $a \ge 0$. The solid obtained by rotating about the y-axis the region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is equal to $2\pi\int_a^b x|f(x) - g(x)|dx$.*

Example 6.9. Consider the solid S obtained by rotating about the x-axis the region $R = \{(x,y)|0 \le x \le 1, 0 \le y \le \sqrt{x}\}$. The volume of S can be computed in two ways. The first one is to consider the cross section of S, which is a disc. This implies

$$V = \int_0^1 \pi x dx = \frac{\pi x^2}{2}\Bigg|_0^1 = \frac{\pi}{2}.$$

The second one is to write $R = \{(x,y)|0 \le y \le 1, y^2 \le x \le 1\}$. By the method of cylindrical shells, we have

$$V = \int_0^1 2\pi y(1 - y^2)dy = 2\pi\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{2}.$$

Example 6.10. Let R be the region bounded by $y = x^2$, $x = 0$, $x = 1$ and the x-axis and let S be the solid obtained by rotating S about $x = 2$. Note that the volume of S is the same as the volume of the solid obtained by rotating about the x-axis the region bounded by $y = (x - 2)^2$, $x = 1$, $x = 2$ and $y = 0$. This implies

$$V = \int_1^2 2\pi x(x - 2)^2 dx = \frac{5\pi}{6}.$$