6.4. **Average value of a function.** (Sec. 6.5 in the textbook.)

For numbers $y_1, ..., y_n$, their average is defined by

$$\frac{y_1 + \cdots + y_n}{n} = y_1 \times \frac{1}{n} + \cdots + y_n \times \frac{1}{n}.$$

For a function $f(x)$ defined on $[a, b]$, we partition $[a, b]$ into n subintervals of equal length, say $[x_{i-1}, x_i]$ for $1 \le i \le n$, and let x_i^* be a sample point in $[x_{i-1}, x_i]$. It is natural to define the average value of f to be the asymptotic average of $f(x_1^*), ..., f(x_n^*)$. Note that if f is integrable on $[a, b]$ and $\Delta x = (b - a)/n$, then

$$\lim_{n\to\infty} \frac{f(x_1^*) + \cdots + f(x_n^*)}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \frac{\Delta x}{b-a} = \frac{1}{b-a} \int_a^b f(x)dx.$$

Definition 6.2. For any function f integrable on $[a, b]$, its *average value* on $[a, b]$ is defined to be $\frac{1}{b-a} \int_a^b f(x)dx$.

Example 6.11. Let $f(x) = x$ and $g(x) = x^2$. The average values of f, g on $[0, 2]$ are

$$\frac{1}{2} \int_0^2 x dx = 1 = f(1), \quad \frac{1}{2} \int_0^2 x^2 dx = \frac{4}{3} = g\left(\frac{2}{\sqrt{3}}\right).$$

Theorem 6.4 (The mean value theorem for integration)**.** *If f is continuous on $[a, b]$, then there is $c \in (a, b)$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx \quad \textit{or equivalently} \quad \int_a^b f(x)dx = f(c)(b-a).$$

Proof. Set $F(x) = \int_a^x f(t)dt$. By the first part of the fundamental theorem of calculus, F is continuous on $[a, b]$, differentiable on (a, b) and $F' = f$. Applying the mean value theorem to F implies that there is $c \in (a, b)$ such that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{1}{b-a} \int_a^b f(t)dt.$$

□