7.1. Integration by parts. (Sec. 7.1 in the textbook.)

The technique of integration by parts is originated from the product rule of differentiation. Integrating both sides of $(fg)' = f'g + fg'$ implies

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Formally, we set $u = f(x)$ and $v = g(x)$. It is easy to see their differentials, $du = f'(x)dx$ and $dv = g'(x)dx$. Immediately, the integration by parts can be restated as $\int udv = uv - \int vdu$.

Example 7.1. Determine (1) $\int x\cos xdx$; (2) $\int x^2e^x dx$; (3) $\int \ln xdx$; (4) $\int e^x \cos xdx$.

For (1), let $f(x) = x$ and $g'(x) = \cos x$. Then, $f'(x) = 1$, $g(x) = \sin x$ and

$$\int x\cos xdx = \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx = x\sin x + \cos x + C.$$

For (2), let $f(x) = x^2$ and $g'(x) = e^x$. Then, $f'(x) = 2x$, $g(x) = e^x$ and

$$\int x^2e^x dx = x^2e^x - 2\int xe^x dx.$$

Again, letting $f(x) = x$ and $g(x) = e^x$ leads to

$$\int xe^x dx = xe^x - \int e^x dx = (x-1)e^x + C.$$

Hence, we have

$$\int x^2e^x dx = (x^2 - 2x + 2)e^x + C.$$

For (3), let $u = \ln x$ and $dv = dx$. Then, $du = (1/x)dx$, $v = x$ and

$$\int \ln xdx = x\ln x - \int x\cdot\frac{1}{x}dx = x(\ln x - 1) + C.$$

For (4), note that

$$\int e^x \cos xdx = \int \cos xd(e^x) = e^x\cos x + \int e^x \sin xdx.$$

and

$$\int e^x \sin xdx = \int \sin xd(e^x) = e^x\sin x - \int e^x \cos xdx.$$

This implies

$$\int e^x\cos xdx = \frac{\sin x + \cos x}{2}e^x + C, \quad \int e^x \sin xdx = \frac{\sin x - \cos x}{2}e^x + C.$$

For the definite integral, if f and g are continuously differentiable on $[a,b]$, then,

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx.$$

Example 7.2. To compute $\int_0^{1/2} \sin^{-1} xdx$, we set $u = \sin^{-1} x$ and $dv = dx$. By the integration by parts, this implies

$$\int_0^{1/2} \sin^{-1} xdx = x\sin^{-1} x\Big|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx = \frac{\pi}{12} + \sqrt{1-x^2}\Big|_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Example 7.3. Consider the indefinite integral $\int \cos^n x dx$. Let $u = \cos^{n-1} x$ and $dv = \cos x dx$. This implies $du = (n-1)\cos^{n-2} x(-\sin x)$, $v = \sin x$ and

$$\begin{aligned}\int \cos^n x dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (\cos^{n-2} x - \cos^n x) dx\end{aligned}$$

This implies

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Example 7.4. To compute $\int \csc x dx$, we write

$$\frac{1}{\sin x} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}.$$

Note that $\frac{d}{dx}\tan\frac{x}{2} = \frac{1}{2}\sec^2\frac{x}{2}$. This implies

$$\int \csc x dx = \int \frac{1}{\sin x} dx = \ln|\tan\tfrac{x}{2}| + C.$$