7.2. **Trigonometric integrals.** (Sec. 7.2 in the textbook)

Consider the following integral.

$$\int \sin^3 x dx.$$

Obviously, the substitution $u = \sin x$ is not helpful because $du = \cos x dx$ will generate a term of $\cos x$. By using the identity of $\sin^2 x + \cos^2 x = 1$, one may rewrite the integrand as $\sin^3 x = (1 - \cos^2 x)\sin x$. Thereafter, the substitution of $u = \cos x$ yields $du = -\sin x dx$ and then

$$\int \sin^3 x dx = \int (1-u^2)(-du) = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

This method applies for integrands $\sin^m x \cos^n x$, where m or n is odd. For integrals of the following type,

$$\int \sin^4 x \cos^6 x dx,$$

one may use the following formulas

$$\sin x \cos x = \frac{1}{2}\sin 2x, \quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

to write

$$\sin^4 x \cos^6 x = \frac{\sin^4 2x}{16} \cdot \frac{1+\cos 2x}{2} = \frac{1}{32}(\sin^4 2x + \sin^4 2x \cos 2x).$$

Clearly, the second term can be integrated by the previous method, while the first term should be rewritten as

$$\sin^4 2x = \left(\frac{1-\cos 4x}{2}\right)^2 = \frac{1 - 2\cos 4x + \cos^2 4x}{4} = \frac{1-2\cos 4x}{4} + \frac{1+\cos 8x}{8}.$$

Strategy to compute $\int \sin^m x \cos^n x dx$ Assume that $n \geq 0$ and $m \geq 0$.

(1) When m is odd, say $m = 2k+1$ with $k \geq 0$, we use the identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$ to get

$$\int \sin^{2k+1} x \cos^n x dx = -\int (1-u^2)^k u^n du\Big|_{u=\cos x}. \tag{7.1}$$

(2) When n is odd, say $n = 2\ell + 1$ with $\ell \geq 0$, we use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$ to get

$$\int \sin^m x \cos^{2\ell+1} dx = \int u^m (1-u^2)^\ell du\Big|_{u=\sin x}. \tag{7.2}$$

(3) When m and n are both even, we use the following identities

$$\sin x \cos x = \frac{1}{2}\sin 2x, \quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2},$$

to rewrite the integrand iteratively until (1) or (2) is applicable.

Remark 7.1. By the binomial theorem, (7.1) and (7.2) become

$$\int \sin^{2k+1} x \cos^n x dx = \sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \frac{(\cos x)^{n+2i+1}}{n+2i+1} + C$$

and

$$\int \sin^m x \cos^{2\ell+1} x dx = \sum_{i=0}^{\ell} \binom{\ell}{i} (-1)^i \frac{(\sin x)^{m+2i+1}}{m+2i+1} + C.$$

Strategy to compute $\int \tan^m x \sec^n x dx$ Assume that $n \geq 0$ and $m \geq 0$.

(1) When m is odd, say $m = 2k+1$ with $k \geq 0$, and $n \geq 1$, we use the identity $\tan^2 x = \sec^2 - 1$ and the substitution $u = \sec x$ to get

$$\int \tan^{2k+1} x \sec^n x dx = \int (u^2-1)^k u^{n-1} du \bigg|_{u=\sec x}.$$

(2) When n is even, say $n = 2\ell$ with $\ell \geq 1$, we use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$ to get

$$\int \tan^m x \sec^{2\ell} x dx = \int u^m (1+u^2)^{\ell-1} du \bigg|_{u=\tan x}.$$

(3) When $n = 2k+1$ and $m = 2\ell$, we use the substitution $u = \sin x$ to get

$$\int \tan^{2\ell} x \sec^{2k+1} x dx = \int \frac{\sin^{2\ell} x}{(1-\sin^2 x)^{k+\ell+1}} \cos x dx = \int \frac{u^{2\ell}}{(1-u^2)^{k+\ell+1}} du \bigg|_{u=\sin x},$$

where we refer the reader to Section 7.4 for the last integration.

Remark 7.2. For the integration of $\tan x$ and $\sec x$, we have

$$\int \tan x dx = \ln|\sec x| + C, \qquad \int \sec x dx = \ln|\tan x + \sec x| dx + C.$$

The first one is given by the identity $\tan x = -(\cos x)'/\cos x$, while the second one uses the following ingenuity,

$$\sec x = \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x} = \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} = \frac{(\tan x + \sec x)'}{\tan x + \sec x}.$$

Remark 7.3. For the integral of $\sin mx \cos nx$, $\sin mx \sin nx$ and $\cos mx \cos nx$, one needs

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$