7.3. **Trigonometric substitution.** (Sec. 7.3 in the textbook)

The following are frequently used trigonometric substitutions.

(1) For $\sqrt{a^2-x^2}$, we set $x = a\sin\theta$ with $\theta \in (-\pi/2, \pi/2)$.
(2) For $\sqrt{a^2+x^2}$, we set $x = a\tan\theta$ with $\theta \in (-\pi/2, \pi/2)$.
(3) For $\sqrt{x^2-a^2}$, we set $x = a\sec\theta$ with $\theta \in (0, \pi/2)$ or $\theta \in (\pi, 3\pi/2)$.

*Example* 7.5. To compute $\int \sqrt{1-x^2}dx$, we set $x = \sin\theta$ with $\theta \in (-\pi/2, \pi/2)$. This implies $dx = \cos\theta d\theta$ and

$$\int \sqrt{1-x^2}dx = \int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

*Example* 7.6. Let $A$ be the area of the region enclosed by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Clearly, $A = \frac{2b}{a}\int_{-a}^{a} \sqrt{a^2-x^2}dx$. Applying the substitution $x = au$, we have

$$A = 2ab\int_{-1}^{1} \sqrt{1-u^2}du = ab(\sin^{-1} u + u\sqrt{1-u^2})\Big|_{-1}^{1} = \pi ab.$$

*Example* 7.7. Consider the integral $\int \frac{1}{x^2\sqrt{x^2+4}}dx$. Let $x = 2\tan\theta$ with $\theta \in (-\pi/2, \pi/2)$. This implies $dx = 2\sec^2\theta d\theta$ and

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta.$$

Through the substitution $u = \sin\theta$, one has $du = \cos\theta d\theta$ and hence

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \frac{1}{4}\int \frac{du}{u^2} = -\frac{1}{4\sin\theta} + C = -\frac{\sec\theta}{4\tan\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C,$$

where the last equality uses the fact $\sqrt{x^2+4} = 2\sec\theta$.

*Example* 7.8. To find $\int \frac{dx}{\sqrt{x^2-a^2}}$ with $a > 0$, we set $x = a\sec\theta$ with $\theta \in (0, \pi/2)$. Then, $dx = a\tan\theta\sec\theta d\theta$ and

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \sec\theta d\theta = \ln|\tan\theta + \sec\theta| + C = \ln|x + \sqrt{x^2-a^2}| + C,$$

where the last equality is a result of the fact $\tan\theta + \sec\theta = \frac{x+\sqrt{x^2-a^2}}{a}$.

*Example* 7.9. Consider the integral $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$. By letting $x = \frac{3}{2}\tan\theta$ with $\theta \in [0, \pi/2)$, we have

$$\begin{aligned}\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx &= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}d\theta \\ &= \frac{3}{16}\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}d\theta - \frac{3}{16}\int_0^{\pi/3} \sin\theta d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}d\theta - \frac{3}{32}.\end{aligned}$$

To see the last integral, we set $u = \cos\theta$. This implies $du = -\sin\theta d\theta$ and then

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}d\theta = -\int_1^{1/2} \frac{du}{u^2} = \frac{1}{u}\Big|_1^{1/2} = 1.$$

Hence, $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx = \frac{3}{32}$.