7.4. **Integration of rational functions by partial fractions.** (Sec. 7.4 in the textbook)

In this section, we consider the indefinite integral of rational functions. Let $P(x)$ and $Q(x)$ be polynomials and $f(x) = P(x)/Q(x)$. First, we write $f(x) = S(x) + R(x)/Q(x)$, where $S(x)$ and $R(x)$ are polynomials and the degree of $R(x)$ is less than the degree of $Q(x)$.

Case 1 $Q(x)$ is a product of distinct linear factors, say

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n).$$

In this case, one may write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}.$$

Example 7.10. To compute $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$, we write $2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2)$ and set

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}.$$

In some computations, one has $2A + B + 2C = 1$, $3A + 2B - C = 2$ and $-2A = -1$, which implies $A = 1/2$, $B = 1/5$ and $C = -1/10$. Consequently, we obtain

$$\begin{aligned}\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}dx &= \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{1}{2x - 1}dx - \frac{1}{10}\int \frac{1}{x + 2}dx \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x - 1| - \frac{1}{10}\ln|x + 2| + C'.\end{aligned}$$

Case 2 $Q(x)$ is a product of linear factors and some of them are repeated, say

$$Q(x) = (a_1x + b_1)^{c_1}(a_2x + b_2)^{c_2}\cdots(a_nx + b_n)^{c_n}.$$

In this case, we write

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{c_1} \frac{k_{1i}}{(a_1x + b_1)^i} + \sum_{i=1}^{c_2} \frac{k_{2i}}{(a_2x + b_2)^i} + \cdots + \sum_{i=1}^{c_n} \frac{k_{ni}}{(a_nx + b_n)^i}.$$

Example 7.11. To compute $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$, we first write

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}.$$

Note that $x^3 - x^2 - x + 1 = (x - 1)^2(x + 1)$. By setting

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1},$$

one has $A + C = 0$, $B - 2C = 4$ and $-A + B + C = 0$, which implies $A = 1$, $B = 2$ and $C = -1$. As a result, this leads to

$$\begin{aligned}\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx &= \frac{x^2}{2} + x + \int \frac{dx}{x - 1} + 2\int \frac{dx}{(x - 1)^2} - \int \frac{dx}{x + 1} \\ &= \frac{x^2}{2} + x + \ln|x - 1| - \frac{2}{x - 1} - \ln|x + 1| + C'\end{aligned}$$

Case 3 $Q(x)$ contains irreducible quadratic factors and none of which is repeated. In this case, $R(x)/Q(x)$ can be written as a summation and each term is of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Write

$$\frac{Ax+B}{ax^2+bx+c} = \frac{A}{2a} \times \frac{2ax+b}{ax^2+bx+c} + \frac{B-Ab/(2a)}{ax^2+bx+c}.$$

By setting $u = x + b/(2a)$ and $d = \sqrt{c/a - b^2/4a^2}$, we have

$$\begin{aligned}\int \frac{Ax+B}{ax^2+bx+c}dx &= \frac{A}{2a}\int \frac{2ax+b}{ax^2+bx+c}dx + \frac{2aB-Ab}{2a^2}\int \frac{1}{u^2+d^2}du \\ &= \frac{A}{2a}\ln|ax^2+bx+c| + \frac{2aB-Ab}{2a^2 d}\tan^{-1}\left(\frac{x+b/(2a)}{d}\right) + C',\end{aligned}$$

where the last equality also uses the fact of $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C'$.

Example 7.12. To find $\int \frac{2x^2-x+4}{x^3+4x}dx$, we write

$$\frac{2x^2-x+4}{x^3+4x} = \frac{Ax+B}{x^2+4} + \frac{C}{x}.$$

This implies $A = 1$, $B = -1$ and $C = 1$, and then

$$\int \frac{2x^2-x+4}{x^3+4x}dx = \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C'.$$

Case 4 $Q(x)$ contains a repeated irreducible quadratic factor, say $(ax^2+bx+c)^r$. In this case, $R(x)/Q(x)$ can be expressed as a sum of

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Example 7.13. Consider the integral $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$. By setting

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$

one has $A = 1$, $B = -1$, $C = -1$, $D = 1$ and $E = 0$. This implies

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C'\end{aligned}$$

Rationalizing substitutions Some integrands can be changed into rational functions using appropriate substitutions. For instance, when the integrand has the term $(g(x))^{1/n}$, the substitution of $u = (g(x))^{1/n}$ might work.

Example 7.14. For $\int \frac{\sqrt{x+4}}{x}dx$, we set $u = \sqrt{x+4}$. Note that

$$\frac{\sqrt{x+4}}{x} = \frac{u}{u^2-4}, \quad du = \frac{1}{2\sqrt{x+4}}dx.$$

This implies

$$\begin{aligned}\int \frac{\sqrt{x+4}}{x}dx &= \int \frac{2u^2}{u^2-4}du = 2\left(\int du + 4\int \frac{du}{u^2-4}\right) \\ &= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C' = 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C'\end{aligned}$$