7.5. **Approximating integrals.** (Sec. 7.7 in the textbook)

Note that the following two integrals,

$$\int_0^1 e^{x^2} dx, \quad \int_{-1}^1 \sqrt{1+x^3} dx,$$

can not be precisely evaluated, because their indefinite integrals are not available. In this case, the only way to determine them is to follow the definition of integration,

$$\int_a^b f(x)dx \approx \sum_{i=1}^n f(x_i^*)\Delta x,$$

and see how fast of the above convergence. Three typical ways of selecting sample points are right endpoints, left endpoints and midpoints, which lead to

$$R_n = \sum_{i=1}^n f(x_i)\Delta x, \quad L_n = \sum_{i=1}^n f(x_{i-1})\Delta x, \quad M_n = \sum_{i=1}^n f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x.$$

In the following, we introduce two other approximations of integrations.

Trapezoidal rule

$$\int_a^b f(x)dx \approx T_n = \Delta x \sum_{i=1}^n \frac{f(x_{i-1})+f(x_i)}{2} = \frac{R_n+L_n}{2},$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Remark 7.4. Note that

$$\begin{aligned} T_n &= \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)] \\ &= R_n + \frac{(b-a)[f(a)-f(b)]}{2n} = L_n + \frac{(b-a)[f(b)-f(a)]}{2n}. \end{aligned}$$

Example 7.15. Consider the integral $\int_1^2 1/x dx$. Note that the value of this integral equals $\ln 2$.

(1) Midpoint rule.

$$M_n = \frac{1}{n}\sum_{i=1}^n \frac{1}{1+(2i-1)/(2n)} = \sum_{i=1}^n \frac{1}{n+i-1/2}.$$

(2) Trapezoidal rule.

$$T_n = \frac{1}{n}\sum_{i=1}^n \frac{1}{2}\left(\frac{1}{1+(i-1)/n} + \frac{1}{1+i/n}\right) = \frac{1}{4n} + \sum_{i=1}^n \frac{1}{n+i}.$$

The following are numerical results for $n = 5$, 10 and 15 with $\ln 2 \approx 0.6931$.

n	L_n	R_n	T_n	M_n
5	0.7456	0.6456	0.6956	0.6919
10	0.7187	0.6687	0.6937	0.6928
15	0.7058	0.6808	0.6933	0.6930

n	E_L	E_R	E_T	E_M
5	-0.0524	0.0475	-0.0024	0.0012
10	-0.0256	0.0243	-0.0006	0.0003
15	-0.0126	0.0123	-0.0001	0.000078

Based on the above outcomes, we can make the following conclusions.

(1) The trapezoidal and midpoint rules are much accurate than the endpoint rules.
(2) The errors in the endpoint rules decrease by a multiple factor of 1/2.
(3) The errors in the trapezoidal and midpoint rules decrease by a multiple factor of 1/4.

Theorem 7.1. *Let E_T and E_M be the errors in the trapezoidal and midpoint rules. Assume that $|f''(x)| \le K$ for $x \in [a,b]$. Then.*

$$|E_T| \le \frac{K(b-a)^3}{12n^2}, \quad |E_M| \le \frac{K(b-a)^3}{24n^2}.$$

Simpson's rule Let $n = 2k$ and, for $1 \le j \le k$, consider the parabola passing

$$(x_{2j-2}, y_{2j-2}), \quad (x_{2j-1}, y_{2j-1}), \quad (x_{2j}, y_{2j}).$$

To see the integral of the region bounded by these parabolas, let $y = Ax^2 + Bx + C$ be the parabola passing (x_0, y_0), (x_1, y_1) and (x_2, y_2). For simplicity, we may set $x_0 = -h$, $x_1 = 0$ and $x_2 = h$. This implies

$$y_0 = Ah^2 - Bh + C, \quad y_1 = C, \quad y_2 = Ah^2 + Bh + C.$$

Note that

$$\int_{-h}^{h} (Ax^2 + Bx + C)dx = \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

As a result, Simpson's rule refers to the following sequence,

$$S_n = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{2k-1}) + f(x_{2k})),$$

where $\Delta x = (b-a)/(2k)$.

Remark 7.5. Note that $S_{2k} = \frac{1}{3}T_k + \frac{2}{3}M_k$.

Example 7.16. Consider the integral $\int_1^2 1/x dx$. Simpson's rule with $n = 10$ gives

$$S_{10} = \frac{1}{30}[f(1) + 4f(1.1) + 2f(1.2) + \cdots + 4f(1.9) + f(2)] \approx 0.693150.$$

Actually, $\int_1^2 1/x dx = \ln 2 \approx 0.693147$ and $|S_{10} - \ln 2| \approx 0.000003$.

Theorem 7.2. *Let E_S be the error induced by Simpson's rule. Assume that $|f^{(4)}(x)| \le K$ for $x \in [a,b]$, then*

$$|E_S| \le \frac{K(b-a)^5}{180n^4}.$$