7.6. **Improper integrals.** (Sec. 7.8 in the textbook)

In this section, we extend the definition of integrals to functions either of which domain is an infinite interval or with infinite discontinuity in [a,b]. In either case, the integral is called an *improper integral*.

Definition 7.1 (Continuous integrands on unbounded domains)**.** Let f be a function and $a, b \in \mathbb{R}$.

(1) When $\int_a^t f(x)dx$ exists for all $t \geq a$ and has a limit as $t \to \infty$, define

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx.$$

(2) When $\int_s^b f(x)dx$ exists for all $s \leq b$ and has a limit as $s \to -\infty$, define

$$\int_{-\infty}^b f(x)dx = \lim_{s\to-\infty} \int_s^b f(x)dx.$$

(3) When $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ exist for some $a \in \mathbb{R}$, define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

Remark 7.6. $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called *convergent* if the corresponding limits exist and called *divergent* otherwise. If $\lim_{t\to\infty} \int_a^t f(x)dx = \pm\infty$, we write $\int_a^\infty f(x)dx = \pm\infty$; if $\lim_{s\to-\infty} \int_s^b f(x)dx = \pm\infty$, we write $\int_{-\infty}^b f(x)dx = \pm\infty$.

Remark 7.7. If $f \geq 0$ on $[a, \infty)$, then the area of the region $S = \{(x, y)|x \geq a,\ 0 \leq y \leq f(x)\}$ is defined to be $\int_a^\infty f(x)dx$.

Example 7.17. To compute $\int_{-\infty}^0 xe^x dx$, note that $\int xe^x dx = (x-1)e^x + C$. Then,

$$\lim_{t\to-\infty} \int_t^0 xe^x dx = \lim_{t\to-\infty} [(1-t)e^t - 1] = -1.$$

Example 7.18. Consider the integral $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$. Note that

$$\int_{-\infty}^0 \frac{1}{1+x^2} dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2} dx = \lim_{t\to-\infty} (-\tan^{-1} t) = \frac{\pi}{2}$$

and

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} dx = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}.$$

This implies $\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \pi$.

Example 7.19. Consider the integral $\int_1^\infty x^p dx$. Note that

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C \quad \text{for } p \neq -1, \qquad \int x^{-1} dx = \ln x + C.$$

This implies

$$\lim_{t\to\infty} \int_1^t x^p dx = \lim_{t\to\infty} \begin{cases} \frac{1}{p+1}(t^{p+1} - 1) & \text{for } p \neq -1, \\ \ln t & \text{for } p = -1. \end{cases} = \begin{cases} -1/(1+p) & \text{for } p < -1, \\ \infty & \text{for } p \geq -1. \end{cases}$$

Definition 7.2 (Discontinuous integrands)**.** Let f be a function.

(1) If f is continuous on $[a, b)$ but discontinuous at b, define

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx.$$

(2) If f is continuous on $(a, b]$ but discontinuous at a, define

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx.$$

In (1) and (2), the improper integral is called *convergent* if the limit exists and *divergent* otherwise. Write $\int_a^b f(x)dx = \pm\infty$ if the limit diverges to $\pm\infty$.

(3) If f has discontinuity at $c \in (a, b)$ and $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 7.20. To evaluate $\int_2^5 \frac{1}{\sqrt{x-2}}dx$, since $\frac{1}{\sqrt{x-2}}$ is continuous on $(2, 5]$ and $\int \frac{1}{\sqrt{x-2}}dx = 2\sqrt{x-2} + C$, we have

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{t \to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}}dx = 2 \lim_{t \to 2^+} (\sqrt{3} - \sqrt{t-2}) = 2\sqrt{3}.$$

Example 7.21. To compute $\int_0^1 \ln x dx$, note that $\ln x$ is continuous on $(0, 1]$ and $\int \ln x dx = x(\ln x - 1) + C$. This implies

$$\int_0^1 \ln x dx = \lim_{t \to 0^+} \int_t^1 \ln x dx = \lim_{t \to 0^+} (-1 - t \ln t + t) = -1.$$

Example 7.22. Consider $\int_0^3 \frac{dx}{x-1}$. Clearly, $\frac{1}{x-1}$ has a discontinuous at 1. Note that

$$\int_1^3 \frac{dx}{x-1} = \lim_{t \to 1^+} (\ln 2 - \ln(t-1)) = \infty, \quad \int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \ln|t-1| = -\infty.$$

This implies that $\int_0^3 \frac{dx}{x-1}$ does not exist.

Theorem 7.3 (Comparison test). *Let f and g be functions continuous on $[a, \infty)$. Assume that $0 \le f \le g$.*

(1) *If $\int_a^\infty g(x)dx$ is convergent, then $\int_a^\infty f(x)dx$ is convergent.*
(2) *If $\int_a^\infty f(x)dx$ is divergent, then $\int_a^\infty g(x)dx$ is divergent.*

Example 7.23. Consider the integrals $\int_0^\infty e^{-x^2}dx$ and $\int_0^\infty xe^x dx$. Note that

$$e^{-x^2} \le e^{-x} \quad \forall x \ge 1, \quad xe^x \ge x \quad \forall x.$$

By the comparison test, $\int_0^\infty e^{-x^2}dx$ is convergent, while $\int_0^\infty xe^x dx$ is divergent.