8.1. **Arc length.** (Sec. 8.1 in the textbook)

Let f be a function on $[a,b]$ and L be the length of the curve, $\{(x,f(x))|a \le x \le b\}$. To determine L, we partition $[a,b]$ into n subintervals of equal length and let L_i be the length of the segment connecting $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$. If f is smooth enough, then

$$L \approx \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + [f(x_i) - f(x_{i-1})]^2},$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.

By the mean value theorem for differentiation, if f is differentiable, then there is $x_i^* \in (x_{i-1}, x_i)$ such that $f(x_i) - f(x_{i-1}) = (x_i - x_{i-1})f'(x_i^*) = \Delta x f'(x_i^*)$. Further, if f' is continuous, we may identify L with

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \sqrt{1 + [f'(x_i^*)]^2} = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Theorem 8.1 (The arc length formula)**.** *If f' is continuous on $[a,b]$, then the length L of the curve $y = f(x)$ with $x \in [a,b]$ is given by*

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Example 8.1. Let L be the length of the curve $y = x^2$ with $x \in [0,1]$. By the arc length formula, $L = \int_0^1 \sqrt{1 + (2x)^2} dx$. Set $x = \frac{1}{2}\tan\theta$ with $\theta \in (-\pi/2, \pi/2)$. This implies $dx = \frac{1}{2}\sec^2\theta d\theta$ and $\int \sqrt{1+4x^2} dx = \frac{1}{2}\int \sec^3\theta d\theta$. By setting $u = \sin\theta$ and the following formula

$$\frac{1}{(1-u^2)^2} = \frac{1}{4(1-u)^2} + \frac{1}{4(1+u)^2} + \frac{1}{4(1-u)} + \frac{1}{4(1+u)},$$

we obtain

$$\int \sqrt{1+4x^2} dx = \frac{u}{4(1-u^2)} + \frac{1}{8} \ln\left|\frac{1+u}{1-u}\right| + C = \frac{2x\sqrt{1+4x^2} + \ln|2x + \sqrt{1+4x^2}|}{4} + C.$$

This implies $L = \frac{\sqrt{5}}{2} + \frac{\ln(2+\sqrt{5})}{4}$.

Definition 8.1. For any continuously differentiable function f, the *arc length function* from $P(a, f(a))$ to $Q(x, f(x))$ with $x > a$ is defined by $s(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt$.

Remark 8.1. In the form of differential, we may write $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ or $(ds)^2 = (dx)^2 + (dy)^2$. Thus, $L = \int ds$. If g is a function of y, the arc length function becomes $ds = \sqrt{1 + (g')^2} dy = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$.

Example 8.2. The arc length function of $f(x) = \frac{e^x + e^{-x}}{2}$ from $(0,1)$ is

$$s(x) = \int_0^x \sqrt{1 + [f'(t)]^2} dt = \int_0^x \sqrt{1 + \left(\frac{e^t - e^{-t}}{2}\right)^2} dt = \frac{1}{2}\int_0^x (e^t + e^{-t}) dt = \frac{e^x - e^{-x}}{2}.$$