8.2. **Area of a surface of revolution.** (Sec. 8.2 in the textbook)

Note that the surface area of a corn of which cutting isosceles triangle has sides length ℓ and bottom length $2r$ is $(\pi\ell^2)\cdot\frac{2\pi r}{2\pi\ell}=\pi r\ell$. If the cone of side length $\ell_1+\ell$ and bottom length $2r_2$ is cut out the top part of which cutting isosceles triangle has side length ℓ_1 and bottom length $2r_1$, then the area of the remaining band is

$$A=\pi r_2(\ell+\ell_1)-\pi r_1\ell_1=\pi r_2\ell+\pi(r_2-r_1)\cdot\frac{r_1\ell}{r_2-r_1}=\pi(r_1+r_2)\ell.$$

Let f be a function which is nonnegative on $[a,b]$ and A be the area of the surface obtained by rotating the curve $y=f(x)$ on $[a,b]$ about the x-axis. By partitioning $[a,b]$ into n subintervals of equal width, one has

$$A\approx\sum_{i=1}^{n}\pi[f(x_{i-1})+f(x_i)]\sqrt{(\Delta x)^2+[f(x_{i-1})-f(x_i)]^2},$$

where $\Delta x=\frac{b-a}{n}$ and $x_i=a+i\Delta x$. Assume that f is differentiable on $[a,b]$. By the mean value theorem, we may choose $x_i^*\in(x_{i-1},x_i)$ such that $\sqrt{(\Delta x)^2+|f(x_{i-1})-f(x_i)|^2}=\sqrt{1+[f'(x_i^*)]^2}\Delta x$. Note that $f(x_{i-1})\approx f(x_i^*)$ and $f(x_i)\approx f(x_i^*)$. In addition with the continuity of f' on $[a,b]$, we obtain

$$A=\lim_{n\to\infty}\sum_{i=1}^{n}\pi[f(x_{i-1})+f(x_i)]\sqrt{(\Delta x)^2+[f(x_{i-1})-f(x_i)]^2}=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx.$$

Theorem 8.2. *If f is non-negative and has continuous derivative on $[a,b]$, then the area of the surface obtained by rotating the curve $y=f(x)$ with $x\in[a,b]$ about the x-axis is given by*

$$A=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

Remark 8.2. If g is nonnegative and continuously differentiable on $[c,d]$, then the area of the surface obtained by rotating the curve $x=g(y)$ with $y\in[c,d]$ about the y-axis is given by

$$A=\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}dy=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy.$$

Remark 8.3. Let s be the arc length function of $f(x)$ or $g(y)$ and A_x,A_y be the areas of the surfaces obtained by rotating $y=f(x)$ and $x=g(y)$ about the x-axis and y-axis. Then, $A_x=\int 2\pi y ds$ and $A_y=\int 2\pi x ds$.

Example 8.3. Let A be the area of the surface obtained by rotating the curve $y=x^2$ with $x\in[1,2]$ about the y-axis. By rewriting $x=y^{1/2}$ with $y\in[1,4]$, we have

$$A=\int 2\pi x ds=\int_1^4 2\pi y^{1/2}\sqrt{1+\frac{1}{4y}}dy=\int_1^2 2\pi x\sqrt{1+4x^2}dx=\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5}).$$

Example 8.4. To compute the area A of the surface of a ball with radius r, let $y=\sqrt{r^2-x^2}$ with $x\in[-r,r]$. Then,

$$A=\int 2\pi y ds=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}dx=\int_{-r}^{r}2\pi r dx=4\pi r^2.$$