A remark on jump conditions for the three-dimensional Navier-Stokes equations involving an immersed moving membrane

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Abstract

Jump conditions for the pressure, the velocity, and their normal derivatives across an immersed moving membrane in an incompressible fluid are derived. The discontinuities are due to the singular forces along the membrane. Instead of using the delta function formulation, those jump conditions can be used to formulate the governing equations in an alternative form. It is also useful for developing more accurate numerical methods such as immersed interface method for the Navier-Stokes equations involving moving interface.

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Equations of motion. Problems of biological fluid mechanics often involve an interaction of a viscous incompressible fluid with an elastic moving membrane. One can consider this membrane as a part of the fluid which exerts forces to the fluid and at the same time moves along with the fluid. The mathematical formulation and numerical method for this kind of problem was first introduced by Peskin to simulate the blood flow through heart valves [5]. This method (immersed boundary method) has been applied successfully to other different biology and human physiology problems. Readers who are interested in this method can refer to [6] and the references therein. The three-dimensional governing equations can be written as follows:

$$\rho\,(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \mu\,\Delta\mathbf{u} + \mathbf{f}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2}$$

$$\mathbf{f}(\mathbf{x},t) = \int_\sigma \mathbf{F}(r,s,t)\,\delta(\mathbf{x}-\mathbf{X}(r,s,t))\,dr\,ds, \tag{3}$$

$$\frac{\partial \mathbf{X}(r,s,t)}{\partial t} = \mathbf{u}(\mathbf{X}(r,s,t),t) = \int_\Omega \mathbf{u}(\mathbf{x},t)\,\delta(\mathbf{x}-\mathbf{X}(r,s,t))\,d\mathbf{x}, \tag{4}$$

$$\mathbf{F}(r,s,t) = \mathbf{S}(\mathbf{X}(r,s,t),t), \tag{5}$$

where $\mathbf{u}(\mathbf{x},t)$ is the fluid velocity, ρ the density, $p(\mathbf{x},t)$ the pressure, μ the viscosity, and $\mathbf{f}(\mathbf{x},t)$ is the force density exerted by the membrane. The membrane σ is represented by $\mathbf{X}(r,s,t)$, where r,s

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are the parameters of some reference configuration with $0 \le r \le L_r, 0 \le s \le L_s$. Eqs. (1)-(2) are the familiar Navier-Stokes equations of a viscous incompressible fluid. Eqs. (3)- (4) represent the interaction between the membrane and the fluid, where the Dirac delta function in these equations is three-dimensional. In Eq. (3), the force density is applied to the fluid by the membrane, while in Eq. (4) the membrane moves at the local fluid velocity (no-slip condition). Without loss of generality, we assume that the membrane force $\mathbf{F}(r, s, t)$ is a function of its configuration in Eq. (5).

In this paper, the delta function formulation described in Eqs. (1)-(5) will be reduced to a standard formulation (without the singular delta force terms in Eq. (1) -(5)) in terms of the jump conditions (16), (17), and (24) which hold across the immersed membrane. The jump conditions not only provide important insight into the solution behavior but also can be used to develop accurate numerical algorithms such as the immersed interface method [2, 4].

Since the membrane force $\mathbf{F}(r, s, t)$ is integrated by a three-dimensional Dirac delta function over a two-dimensional surface σ, the resulting force density $\mathbf{f}(\mathbf{x}, t)$ is a one-dimensional singular Dirac delta function. Mathematically, $\mathbf{f}(\mathbf{x}, t)$ can be viewed as a distribution function whose action on any test function $\mathbf{w}(\mathbf{x}, t)$ is defined by

$$
\begin{aligned}
\langle \mathbf{f}, \mathbf{w} \rangle &= \int_\Omega \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{w}(\mathbf{x}, t)\, d\mathbf{x} \\
&= \int_\Omega \int_0^{L_r} \int_0^{L_s} \mathbf{F}(r, s, t) \delta(\mathbf{x} - \mathbf{X}(r, s, t))\, \mathbf{w}(\mathbf{x}, t)\, dr\, ds \cdot d\mathbf{x} \\
&= \int_0^{L_r} \int_0^{L_s} \mathbf{F}(r, s, t) \cdot \int_\Omega \mathbf{w}(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{X}(r, s, t))\, d\mathbf{x}\, dr\, ds \\
&= \int_0^{L_r} \int_0^{L_s} \mathbf{F}(r, s, t) \cdot \mathbf{w}(\mathbf{X}(r, s, t), t)\, dr\, ds.
\end{aligned}
$$

In particular, if we choose $\mathbf{w}(\mathbf{x}, t)$ to be the velocity $\mathbf{u}(\mathbf{x}, t)$, then the above equality means that the total work done by the membrane is equal to the total work done on the fluid.

As we mentioned before, the formulation given in Eq. (1) involves a one-dimensional delta function, one can expect that the velocity and pressure will not be smooth across the membrane. In this paper, we shall derive the jump conditions analytically across the membrane. While there are similar results as in Theorem 1 in the literature, for example, [3, 7], the results in Theorem 2 in this paper are unique. The jump conditions are not very much different in two and three dimensions, however the proof of three dimensional results is substantial different from that in two dimensions.

Jump conditions across the membrane. Eq. (4) means that the membrane moves at local fluid velocity; thus, the velocity field is continuous across the membrane. That is,

$$[\mathbf{u}] = \mathbf{0}, \tag{6}$$

where [] represents a jump across the membrane. Throughout this paper, we assume the membrane surface is smooth so that any unit normal $\mathbf{n}$ to the membrane can be defined by

$$\mathbf{n} = \frac{\partial \mathbf{X}/\partial r \times \partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial r \times \partial \mathbf{X}/\partial s|}. \tag{7}$$

Lemma 1

$$
\begin{aligned}
[\mathbf{u}_t] + [\nabla \mathbf{u}] \cdot \mathbf{u} &= \mathbf{0}, && (8) \\
\left[\frac{\partial \mathbf{u}}{\partial \mathbf{n}}\right] \cdot \mathbf{n} &= 0. && (9)
\end{aligned}
$$

Proof: To prove Eq. (8), we simply differentiate Eq. (6) with respect to t. Thus we have

$$\begin{aligned}\left[\mathbf{u}_t + \nabla\mathbf{u}\cdot\frac{\partial\mathbf{X}}{\partial t}\right] &= [\mathbf{u}_t + \nabla\mathbf{u}\cdot\mathbf{u}] \\ &= [\mathbf{u}_t] + [\nabla\mathbf{u}]\cdot\mathbf{u} \\ &= \mathbf{0}.\end{aligned}$$

To prove the second equality, we introduce the following local coordinates:

$$\xi = (x-X)\cos\alpha_x + (y-Y)\cos\beta_x + (z-Z)\cos\gamma_x, \tag{10}$$

$$\eta = (x-X)\cos\alpha_y + (y-Y)\cos\beta_y + (z-Z)\cos\gamma_y, \tag{11}$$

$$\zeta = (x-X)\cos\alpha_z + (y-Y)\cos\beta_z + (z-Z)\cos\gamma_z. \tag{12}$$

Here ξ is pointing to the normal direction of the surface, and the membrane surface can be expressed as

$$\xi = \chi(\eta,\zeta), \qquad \text{with} \qquad \chi(0,0)=0, \;\; \chi_\eta(0,0)=0, \;\; \chi_\zeta(0,0)=0, \tag{13}$$

where η and ζ are pointing to two orthogonal directions in the tangent plane of the membrane.

Applying those coordinates to the incompressibility condition

$$[u_x + v_y + w_z] = 0, \tag{14}$$

we can get

$$\begin{aligned}&[u_\xi]\cos\alpha_x + [v_\xi]\cos\beta_x + [w_\xi]\cos\gamma_x + [u_\eta]\cos\alpha_y + [u_\zeta]\cos\alpha_z + \\ &\quad [v_\eta]\cos\beta_y + [v_\zeta]\cos\beta_z + [w_\eta]\cos\gamma_y + [w_\zeta]\cos\gamma_z \\ &\quad = [u_\xi]\cos\alpha_x + [v_\xi]\cos\beta_x + [w_\xi]\cos\gamma_x \equiv 0,\end{aligned}$$

since $[u_\eta]$, $[u_\zeta]$, $[v_\eta]$, $[v_\zeta]$, $[w_\eta]$, and $[w_\zeta]$ are all zero. Thus, we have

$$\left[\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right]\cdot\mathbf{n} = [u_\xi]\cos\alpha_x + [v_\xi]\cos\beta_x + [w_\xi]\cos\gamma_x = 0. \tag{15}$$

Theorem 1 *The pressure and the velocity normal derivatives across the membrane satisfy*

$$[p] = \frac{\mathbf{F}\cdot\mathbf{n}}{|\partial\mathbf{X}/\partial r \times \partial\mathbf{X}/\partial s|}, \tag{16}$$

$$\mu\left[\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right] = \frac{(\mathbf{F}\cdot\mathbf{n})\,\mathbf{n} - \mathbf{F}}{|\partial\mathbf{X}/\partial r \times \partial\mathbf{X}/\partial s|}. \tag{17}$$

Proof: Let us choose a banded domain $\Omega_\epsilon(t)$ enclosing the immersed membrane $\sigma(t)$ with the outer and inner membrane $\sigma_\epsilon^+(t)$ and $\sigma_\epsilon^-(t)$. Here we denote ϵ as a small distance parameter from $\sigma(t)$ to $\sigma_\epsilon^+(t)$ and to $\sigma_\epsilon^-(t)$. Suppose $\phi(\mathbf{x})$ be any smooth function with compact support on $\Omega_\epsilon(t)$. Multiplying $\phi(\mathbf{x})$ on both sides of Eq. (1) and integrating over $\Omega_\epsilon(t)$, we obtain

$$\begin{aligned}&\int_{\Omega_\epsilon(t)} \rho\,(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u})\,\phi\,d\mathbf{x} + \int_{\Omega_\epsilon(t)} \nabla p\,\phi\,d\mathbf{x} \\ = &\int_{\Omega_\epsilon(t)} \mu\,\Delta\mathbf{u}\,\phi\,d\mathbf{x} + \int_{\Omega_\epsilon(t)}\int_0^{L_r}\int_0^{L_s} \mathbf{F}(r,s,t)\,\delta(\mathbf{x}-\mathbf{X}(r,s,t))\,ds\,\phi\,d\mathbf{x}.\end{aligned}$$

Applying the transport theorem [1] to the first term and using the fact of $\Omega_\epsilon(t)$ enclosing $\sigma(t)$ in the last term, the above equation becomes

$$\frac{d}{dt}\int_{\Omega_\epsilon(t)} \rho\,\mathbf{u}\,\phi\,d\mathbf{x} + \int_{\Omega_\epsilon(t)} \nabla p\,\phi\,d\mathbf{x} = \int_{\Omega_\epsilon(t)} \mu\,\Delta\mathbf{u}\,\phi\,d\mathbf{x} + \int_0^{L_r}\int_0^{L_s} \mathbf{F}(r,s,t)\,\phi(\mathbf{X}(r,s,t))\,dr\,ds. \quad (18)$$

Now consider the x-component of the previous equation in the following,

$$\frac{d}{dt}\int_{\Omega_\epsilon(t)} \rho\,u_1\,\phi\,d\mathbf{x} + \int_{\Omega_\epsilon(t)} p_x\,\phi\,d\mathbf{x} = \int_{\Omega_\epsilon(t)} \mu\,\Delta u_1\,\phi\,d\mathbf{x} + \int_0^{L_r}\int_0^{L_s} F_1(r,s,t)\,\phi(\mathbf{X}(r,s,t))\,dr\,ds, \quad (19)$$

where the subscript 1 represents the x-component of variables. Since u_1 is continuous and bounded, the first term of Eq. (19) tends to zero as $\epsilon \to 0$. The second term of Eq.(19) becomes

$$\begin{aligned}
\int_{\Omega_\epsilon(t)} p_x\,\phi\,d\mathbf{x} &= \int_{\Omega_\epsilon(t)} \nabla\cdot(p,0,0)\,\phi\,d\mathbf{x} \\
&= \int_{\sigma_\epsilon^+} p\,n_1\,\phi\,d\ell - \int_{\sigma_\epsilon^-} p\,n_1\,\phi\,d\ell - \int_{\Omega_\epsilon(t)} p\,\phi_x\,d\mathbf{x} \\
&\to \int_\sigma [p]\,n_1\,\phi\,d\ell \qquad \text{as } \epsilon \to 0 \text{ (because } p \text{ is bounded)} \\
&= \int_0^{L_r}\int_0^{L_s} [p]\,n_1\,\phi\,\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right|\,dr\,ds.
\end{aligned}$$

The third term of Eq.(19) is

$$\begin{aligned}
\int_{\Omega_\epsilon(t)} \mu\,\Delta u_1\,\phi\,d\mathbf{x} &= \int_{\sigma_\epsilon^+} \mu\,\frac{\partial u_1}{\partial\mathbf{n}}\,\phi\,d\ell - \int_{\sigma_\epsilon^-} \mu\,\frac{\partial u_1}{\partial\mathbf{n}}\,\phi\,d\ell - \int_{\Omega_\epsilon(t)} \mu\,\nabla u_1\cdot\nabla\phi\,d\mathbf{x} \\
&\to \int_\sigma \mu\left[\frac{\partial u_1}{\partial\mathbf{n}}\right]\phi\,d\ell \qquad \text{as } \epsilon \to 0 \text{ (because } \nabla u_1 \text{ is bounded)} \\
&= \int_0^{L_r}\int_0^{L_s} \mu\left[\frac{\partial u_1}{\partial\mathbf{n}}\right]\phi\,\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right|\,dr\,ds.
\end{aligned}$$

Since ϕ is arbitrary, we derive

$$[p]\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right| n_1 = \mu\left[\frac{\partial u_1}{\partial\mathbf{n}}\right]\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right| + F_1. \quad (20)$$

Similarly, we can derive the relation for y,z components. Thus, we have

$$[p]\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right|\mathbf{n} = \mu\left[\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right]\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right| + \mathbf{F}. \quad (21)$$

Taking inner product of Eq. (21) with $\mathbf{n}$, and using the result of Eq. (9), we can easily obtain Eq. (16). Substitute this pressure jump into Eq. (21), we have

$$\mu\left[\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right]\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right| = (\mathbf{F}\cdot\mathbf{n})\,\mathbf{n} - \mathbf{F}. \quad (22)$$

Therefore, the proof is complete.

The pressure jump condition in (16) can be understood most easily by considering an inflated balloon in equilibrium with its surroundings. In this scenario, the pressure jump across the elastic membrane is exactly balanced by the elastic forces normal to the surface of the membrane.

The jump in momentum flux across the membrane, given by (17) can be interpreted by considering a case in which only the normal component of force at a membrane is non-zero. In this case, there is no viscous shear stress at the membrane, and the jump in equation (17) is exactly zero. The membrane does not serve as a source (or sink) of momentum. The momentum flux jump only depends upon the tangential forces at the boundary.

Theorem 2 *Let both independent unit vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ on the tangent plane of the membrane denote by*

$$\boldsymbol{\tau}_1 = \frac{\partial \mathbf{X}/\partial r}{|\partial \mathbf{X}/\partial r|}, \qquad \boldsymbol{\tau}_2 = \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|}. \tag{23}$$

Under the assumption of $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = 0$, the normal derivative of the pressure across the membrane satisfies

$$\left[\frac{\partial p}{\partial \mathbf{n}}\right] \left|\frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial s}\right| = \frac{\partial}{\partial r}\left(\frac{\mathbf{F}\cdot\boldsymbol{\tau}_1}{|\partial \mathbf{X}/\partial r|}\right) + \frac{\partial}{\partial s}\left(\frac{\mathbf{F}\cdot\boldsymbol{\tau}_2}{|\partial \mathbf{X}/\partial s|}\right). \tag{24}$$

Proof: Applying the divergence operator to both side of the Navier-Stokes equations Eq. (1), and using the incompressibility constraint Eq. (2), we have

$$\nabla \cdot \left\{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)\right\} + \Delta p = \nabla \cdot \mathbf{f}. \tag{25}$$

Multiplying ϕ on both sides of Eq. (25) and integrating over $\Omega_\epsilon(t)$, we obtain

$$\int_{\Omega_\epsilon(t)} \nabla \cdot \left\{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)\right\} \phi\, d\mathbf{x} + \int_{\Omega_\epsilon(t)} \Delta p\, \phi\, d\mathbf{x} = \int_{\Omega_\epsilon(t)} \nabla \cdot \mathbf{f}\, \phi\, d\mathbf{x}. \tag{26}$$

The first term of Eq. (26) is

$$\begin{aligned}
& \int_{\Omega_\epsilon(t)} \nabla \cdot \left\{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right)\right\} \phi\, d\mathbf{x} \\
= & \int_{\sigma_\epsilon^+ \cup \sigma_\epsilon^-} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) \phi \cdot \mathbf{n}\, d\ell - \int_{\Omega_\epsilon(t)} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) \cdot \nabla\phi\, d\mathbf{x} \\
= & \int_{\sigma_\epsilon^+ \cup \sigma_\epsilon^-} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) \phi\ \cdot \mathbf{n}\, d\ell - \frac{d}{dt}\int_{\Omega_\epsilon(t)} \rho\, \mathbf{u}\ \cdot \nabla\phi\, d\mathbf{x} \\
\to & \ 0 \qquad \text{as } \epsilon \to 0, \text{(using Eq. (8) and } \mathbf{u} \text{ is bounded).}
\end{aligned}$$

The second term of Eq. (26) becomes

$$\begin{aligned}
\int_{\Omega_\epsilon(t)} \Delta p\, \phi\, d\mathbf{x} &= \int_{\Omega_\epsilon(t)} \Delta\phi\, p\, d\mathbf{x} + \int_{\sigma_\epsilon^+ \cup \sigma_\epsilon^-} \phi\, \frac{\partial p}{\partial \mathbf{n}}\, d\ell - \int_{\sigma_\epsilon^+ \cup \sigma_\epsilon^-} p\, \frac{\partial \phi}{\partial \mathbf{n}}\, d\ell \\
&\to \int_\sigma \left[\frac{\partial p}{\partial \mathbf{n}}\right] \phi\, d\ell - \int_\sigma [p]\, \frac{\partial \phi}{\partial \mathbf{n}}\, d\ell \qquad \text{as } \epsilon \to 0, \text{(because } p \text{ is bounded).} \\
&= \int_0^{L_r}\int_0^{L_s} \left[\frac{\partial p}{\partial \mathbf{n}}\right] \left|\frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial s}\right| \phi\, dr\, ds - \int_0^{L_r}\int_0^{L_s} [p] \left|\frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial s}\right| \frac{\partial \phi}{\partial \mathbf{n}}\, dr\, ds.
\end{aligned}$$

Using the definition of the dipole distribution(the generalized derivative of the Dirac delta function), the last term of Eq. (26) can be written as

$$\begin{aligned}
\int_{\Omega_\epsilon(t)} \nabla\cdot\mathbf{f}\,\phi\,d\mathbf{x} &= \int_{\Omega_\epsilon(t)}\int_0^{L_r}\int_0^{L_s} \nabla\cdot(\mathbf{F}\delta(\mathbf{x}-\mathbf{X}(r,s,t))\,\phi\,dr\,ds\,d\mathbf{x} \\
&= -\int_0^{L_r}\int_0^{L_s} \mathbf{F}\cdot\nabla\phi\,dr\,ds \\
&= -\int_0^{L_r}\int_0^{L_s} \left(\mathbf{F}\cdot\mathbf{n}\,\frac{\partial\phi}{\partial\mathbf{n}} + \mathbf{F}\cdot\boldsymbol{\tau}_1\,\frac{\partial\phi}{\partial\boldsymbol{\tau}_1} + \mathbf{F}\cdot\boldsymbol{\tau}_2\,\frac{\partial\phi}{\partial\boldsymbol{\tau}_2}\right) dr\,ds \\
&= -\int_0^{L_r}\int_0^{L_s} \left(\mathbf{F}\cdot\mathbf{n}\,\frac{\partial\phi}{\partial\mathbf{n}} + \frac{\mathbf{F}\cdot\boldsymbol{\tau}_1}{|\partial\mathbf{X}/\partial r|}\,\frac{\partial\phi}{\partial r} + \frac{\mathbf{F}\cdot\boldsymbol{\tau}_2}{|\partial\mathbf{X}/\partial s|}\,\frac{\partial\phi}{\partial s}\right) dr\,ds \\
&= -\int_0^{L_r}\int_0^{L_s} \mathbf{F}\cdot\mathbf{n}\,\frac{\partial\phi}{\partial\mathbf{n}}\,dr\,ds + \int_0^{L_r}\int_0^{L_s}\left(\frac{\partial}{\partial r}\left(\frac{\mathbf{F}\cdot\boldsymbol{\tau}_1}{|\partial\mathbf{X}/\partial r|}\right) + \frac{\partial}{\partial s}\left(\frac{\mathbf{F}\cdot\boldsymbol{\tau}_2}{|\partial\mathbf{X}/\partial s|}\right)\right)\phi\,dr\,ds.
\end{aligned}$$

Using the equality of Eq. (16) and ϕ is arbitrary, we can easily conclude that the normal derivative jump for the pressure satisfies

$$\left[\frac{\partial p}{\partial\mathbf{n}}\right]\left|\frac{\partial\mathbf{X}}{\partial r}\times\frac{\partial\mathbf{X}}{\partial s}\right| = \frac{\partial}{\partial r}\left(\frac{\mathbf{F}\cdot\boldsymbol{\tau}_1}{|\partial\mathbf{X}/\partial r|}\right) + \frac{\partial}{\partial s}\left(\frac{\mathbf{F}\cdot\boldsymbol{\tau}_2}{|\partial\mathbf{X}/\partial s|}\right). \tag{27}$$

Thus, the proof is complete.

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