



# A short note on Navier–Stokes flows with an incompressible interface and its approximations



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## ABSTRACT

In biological applications, a cell membrane consisting of a lipid bilayer usually behaves as fluid-like interface with surface incompressibility. Here we consider a mathematical formulation for an incompressible interface immersed in Navier–Stokes flows and study the mathematical and physical features for this incompressible interface. The model formulation introduces an unknown tension which acts as a Lagrange’s multiplier to enforce such surface incompressibility. In this note, we show that the spreading operator of the tension and the surface divergence operator of the velocity are skew-adjoint with each other which indicates physically that the tension does not do extra work to the fluid under the condition of surface incompressibility. In order to avoid solving the unknown tension to enforce the surface incompressibility, we adopt a nearly surface incompressible approach (or penalty approach) by introducing two different modified elastic tensions which can be used efficiently in practical numerical simulations. Furthermore, we show that the resultant modified elastic forces have the same mathematical form as the original one derived from the unknown tension.

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## 1. Introduction

In cell biology, a vesicle is a small liquid droplet with a radius of about 10  $\mu\text{m}$  enclosed by a lipid bilayer membrane suspended in a viscous incompressible fluid media. This membrane is negligibly thin with the thickness around 6 nm and exhibits resistance against membrane area dilation and bending. Therefore, it is natural to consider this fluid-like membrane as an incompressible interface  $\Sigma(t)$  suspended in a three-dimensional fluid domain  $\Omega$ . Here, for simplicity, we disregard the membrane bending effect and focus on the tension effect due to surface incompressibility to the fluid. A similar formulation for the deformation of liquid capsules with incompressible interfaces in Stokes flow was considered in [1]. A complete mathematical

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formulation of vesicle modeling consisting of incompressible Navier–Stokes flows with bending and elastic forces acting on the vesicle membrane simultaneously can be found in [2]. Using the immersed boundary formulation [3], the Navier–Stokes flow interacting with an incompressible interface can be written as follows.

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u} + \int_{\Sigma} \mathbf{F}_{\sigma}(\alpha, \beta, t) \delta(\mathbf{x} - \mathbf{X}(\alpha, \beta, t)) dA \quad \text{in } \Omega, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (2)$$

$$\mathbf{F}_{\sigma}(\alpha, \beta, t) = (\nabla_s \sigma - 2\sigma H \mathbf{n})(\alpha, \beta, t) \quad \text{on } \Sigma, \quad (3)$$

$$\frac{\partial \mathbf{X}}{\partial t}(\alpha, \beta, t) = \mathbf{U}(\alpha, \beta, t) = \int_{\Omega} \mathbf{u}(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{X}(\alpha, \beta, t)) d\mathbf{x} \quad \text{on } \Sigma, \quad (4)$$

$$\nabla_s \cdot \mathbf{U} = 0 \quad \text{on } \Sigma. \quad (5)$$

Here, we assume that the fluids inside and outside of the membrane have the same density  $\rho$  and viscosity  $\mu$ . Eqs. (1) and (2) are the incompressible Navier–Stokes equations with the fluid velocity  $\mathbf{u}(\mathbf{x}, t)$  and the pressure  $p(\mathbf{x}, t)$ , where  $\mathbf{x} = (x, y, z)$  denotes Cartesian coordinates in  $\Omega$  and  $t$  is the time. The membrane is assumed as a sufficiently differentiable surface represented by  $\Sigma(t) = \{\mathbf{X}(\alpha, \beta, t) | 0 \leq \alpha \leq \ell_{\alpha}, 0 \leq \beta \leq \ell_{\beta}\}$ , where  $\alpha$  and  $\beta$  are Lagrangian coordinates of the reference configuration. The last term in Eq. (1) is the forcing term arising from the membrane elastic force  $\mathbf{F}_{\sigma}$  and represents the spreading of Lagrangian force  $\mathbf{F}_{\sigma}$  to the Eulerian fluid via the linkage of the Dirac delta function  $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ . Similarly, the membrane surface  $\Sigma(t)$  moves with the Lagrangian velocity  $\mathbf{U}$  which is also interpolated using the local fluid velocity via the delta function in Eq. (4). Since the rate of change of the local stretching factor satisfies

$$\frac{\partial}{\partial t} (|\mathbf{X}_{\alpha} \times \mathbf{X}_{\beta}|) = (\nabla_s \cdot \mathbf{U}) |\mathbf{X}_{\alpha} \times \mathbf{X}_{\beta}| \quad (6)$$

as shown in [2], the surface incompressibility indicates the above rate of change should equal to zero locally which results in the condition of Eq. (5).

The membrane force  $\mathbf{F}_{\sigma}$  written in Eq. (3) can be derived by taking the variational derivative of the elastic energy  $E_{\sigma} = \int_{\Sigma(t)} \sigma(\alpha, \beta, t) dA$  with respect to the configuration in which  $\sigma$  is the variable membrane tension. We shall have the similar derivation later on. Unlike the pure droplet problem, the surface tension  $\sigma$  in above formulation is not a physical quantity but is self-adjustable and acts as a Lagrange’s multiplier to enforce the surface incompressibility condition Eq. (5) which plays the same role as the pressure  $p$  in Navier–Stokes equations to enforce the fluid incompressibility of Eq. (2). In Eq. (3),  $H$  is the mean curvature and  $\mathbf{n}$  is the outward unit normal vector of the membrane surface. The detailed expressions for the surface gradient  $\nabla_s \sigma$  and the surface divergence  $\nabla_s \cdot \mathbf{U}$  in terms of the coefficients of the first fundamental form will be given in next section.

The contribution of this paper is twofold. Firstly, we show that the spreading operator of the tension and the surface divergence operator of the velocity are skew-adjoint with each other which indicates physically that the tension does not do extra work to the fluid under the condition of surface incompressibility. Secondly, to avoid solving the unknown tension to enforce the surface incompressibility, we adopt a nearly surface incompressible approach (or penalty approach) which is shown quite useful in practical numerical simulations [2,4]. We introduce two other elastic penalty energies which are different from the one in our previous work [2,4] whose resultant elastic forces turn out to have the same mathematical form as in Eq. (3). Notice that, the present penalty energy is derived using classical differential geometry while the existing model such as Skalak formulation [5] is derived from strain energy in mechanics. Furthermore, in practice, the numerical scheme based on our present approach does not require to compute two stretching factors defined in orthogonal directions on the surface so the implementation of our approach using local surface areas is simpler than other schemes.

## 2. Surface incompressibility in flows

In this section, we shall prove that the spreading operator of the tension and the surface divergence operator of the velocity are skew-adjoint mathematically. Before to proceed, we first review some preliminaries in classical differential geometry and express the surface gradient  $\nabla_s \sigma$  and the surface divergence  $\nabla_s \cdot \mathbf{U}$  in terms of the coefficients of the first fundamental form. We then derive alternate formulas for above quantities.

Let us denote two linearly independent tangent vectors on the surface by  $\mathbf{X}_\alpha = \frac{\partial \mathbf{X}}{\partial \alpha}$  and  $\mathbf{X}_\beta = \frac{\partial \mathbf{X}}{\partial \beta}$ , respectively; and assume that  $\mathbf{n} = (\mathbf{X}_\alpha \times \mathbf{X}_\beta) / |\mathbf{X}_\alpha \times \mathbf{X}_\beta|$  is the unit normal vector pointing outward. Throughout the paper, the subscripts of a function denote its partial derivatives. Using these notations, we define the coefficients of the first fundamental form by

$$E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha, \quad F = \mathbf{X}_\alpha \cdot \mathbf{X}_\beta, \quad G = \mathbf{X}_\beta \cdot \mathbf{X}_\beta,$$

and those of the second fundamental form by

$$L = -\mathbf{X}_\alpha \cdot \mathbf{n}_\alpha = \mathbf{X}_{\alpha\alpha} \cdot \mathbf{n}, \quad M = -\mathbf{X}_\alpha \cdot \mathbf{n}_\beta = -\mathbf{X}_\beta \cdot \mathbf{n}_\alpha = \mathbf{X}_{\alpha\beta} \cdot \mathbf{n}, \quad N = -\mathbf{X}_\beta \cdot \mathbf{n}_\beta = \mathbf{X}_{\beta\beta} \cdot \mathbf{n}.$$

Thus, the local stretching factor can be written as  $|\mathbf{X}_\alpha \times \mathbf{X}_\beta| = \sqrt{EG - F^2}$ . The surface gradient operator [6] for a scalar function  $\sigma$  is expressed by

$$\nabla_s \sigma = \frac{GX_\alpha - FX_\beta}{EG - F^2} \sigma_\alpha + \frac{EX_\beta - FX_\alpha}{EG - F^2} \sigma_\beta \quad (7)$$

and similarly, the surface divergence operator for a vector function  $\mathbf{U}$  is

$$\nabla_s \cdot \mathbf{U} = \frac{GX_\alpha - FX_\beta}{EG - F^2} \cdot \mathbf{U}_\alpha + \frac{EX_\beta - FX_\alpha}{EG - F^2} \cdot \mathbf{U}_\beta. \quad (8)$$

The mean curvature  $H = \frac{1}{2} \nabla_s \cdot \mathbf{n}$  can also be written in terms of those coefficients as  $H = \frac{-GL - EN + 2FM}{2(EG - F^2)}$ .

One can also rewrite Eqs. (7) and (8), without using the coefficients of first fundamental form. By substituting  $\mathbf{n} = (\mathbf{X}_\alpha \times \mathbf{X}_\beta) / |\mathbf{X}_\alpha \times \mathbf{X}_\beta|$  and using the vector triple product formulas  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$  and  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$ , we can easily obtain

$$\begin{aligned} \mathbf{X}_\beta \times \mathbf{n} &= \mathbf{X}_\beta \times \frac{\mathbf{X}_\alpha \times \mathbf{X}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} = \frac{(\mathbf{X}_\beta \cdot \mathbf{X}_\beta)\mathbf{X}_\alpha - (\mathbf{X}_\beta \cdot \mathbf{X}_\alpha)\mathbf{X}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} = \frac{GX_\alpha - FX_\beta}{\sqrt{EG - F^2}}, \\ \mathbf{n} \times \mathbf{X}_\alpha &= \frac{\mathbf{X}_\alpha \times \mathbf{X}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} \times \mathbf{X}_\alpha = \frac{(\mathbf{X}_\alpha \cdot \mathbf{X}_\alpha)\mathbf{X}_\beta - (\mathbf{X}_\alpha \cdot \mathbf{X}_\beta)\mathbf{X}_\alpha}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} = \frac{EX_\beta - FX_\alpha}{\sqrt{EG - F^2}}. \end{aligned}$$

Therefore, the surface gradient of  $\sigma$  in Eq. (7) and the surface divergence of  $\mathbf{U}$  in Eq. (8) can be respectively expressed by

$$\nabla_s \sigma = \frac{(\mathbf{X}_\beta \times \mathbf{n})\sigma_\alpha + (\mathbf{n} \times \mathbf{X}_\alpha)\sigma_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}, \quad \nabla_s \cdot \mathbf{U} = \frac{(\mathbf{X}_\beta \times \mathbf{n}) \cdot \mathbf{U}_\alpha + (\mathbf{n} \times \mathbf{X}_\alpha) \cdot \mathbf{U}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}. \quad (9)$$

The following lemma shows the forms of mean curvature vector  $H\mathbf{n}$ , and the elastic force  $\mathbf{F}_\sigma$  in Eq. (3).

### Lemma 1.

$$\begin{aligned} \text{(i)} \quad -2H\mathbf{n} &= \frac{\mathbf{X}_\beta \times \mathbf{n}_\alpha + \mathbf{n}_\beta \times \mathbf{X}_\alpha}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}, \\ \text{(ii)} \quad \nabla_s \sigma - 2\sigma H\mathbf{n} &= \frac{(\sigma(\mathbf{X}_\beta \times \mathbf{n}))_\alpha + (\sigma(\mathbf{n} \times \mathbf{X}_\alpha))_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}. \end{aligned}$$

**Proof.** From  $\mathbf{n} \cdot \mathbf{n} = 1$ , we have  $\mathbf{n}_\alpha \cdot \mathbf{n} = \mathbf{n}_\beta \cdot \mathbf{n} = 0$  meaning that  $\mathbf{n}_\alpha$  and  $\mathbf{n}_\beta$  are both tangent vectors on the surface. So the above vectors can be written in the form of Weingarten equations [6] as  $\mathbf{n}_\alpha = -S_{11}\mathbf{X}_\alpha - S_{21}\mathbf{X}_\beta$  and  $\mathbf{n}_\beta = -S_{12}\mathbf{X}_\alpha - S_{22}\mathbf{X}_\beta$ . Using these representations, we have  $\mathbf{X}_\beta \times \mathbf{n}_\alpha = S_{11}\mathbf{X}_\alpha \times \mathbf{X}_\beta$  and  $\mathbf{n}_\beta \times \mathbf{X}_\alpha = S_{22}\mathbf{X}_\alpha \times \mathbf{X}_\beta$  which lead to  $\mathbf{X}_\beta \times \mathbf{n}_\alpha + \mathbf{n}_\beta \times \mathbf{X}_\alpha = (S_{11} + S_{22})\mathbf{X}_\alpha \times \mathbf{X}_\beta$ . This actually completes the proof of (i), since  $H = -\text{trace}(S_{ij})/2$ . By simply combining the identity in (i) and the surface gradient of  $\nabla_s \sigma$  in Eq. (9), one can easily derive the equality of (ii).  $\square$

In the following theorem, we show that the spreading operator of the tension and the surface divergence operator of the velocity are skew-adjoint with each other. To proceed, let us denote the inner product of two scalar functions on the surface  $\Sigma$  by  $\langle f, g \rangle_\Sigma = \int_\Sigma f(\mathbf{X})g(\mathbf{X}) dA$  and the inner product of two vector functions in the domain  $\Omega$  by  $\langle \mathbf{u}, \mathbf{v} \rangle_\Omega = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x}$ .

**Theorem 1.** *Let us denote the spreading operator of the tension by  $\mathbf{S}[\mathbf{F}_\sigma] = \int_\Sigma (\nabla_s \sigma - 2\sigma H \mathbf{n}) \delta(\mathbf{x} - \mathbf{X}(\alpha, \beta, t)) dA$ . Then the rate of change of work done by the elastic force,  $\frac{dW}{dt}$ , should be equal to  $\frac{dW}{dt} = \langle \mathbf{S}[\mathbf{F}_\sigma], \mathbf{u} \rangle_\Omega = -\langle \sigma, \nabla_s \cdot \mathbf{U} \rangle_\Sigma$ .*

**Proof.**

$$\begin{aligned}
 \frac{dW}{dt} &= \langle \mathbf{S}[\mathbf{F}_\sigma], \mathbf{u} \rangle_\Omega = \int_\Omega \mathbf{S}[\mathbf{F}_\sigma] \cdot \mathbf{u}(\mathbf{x}) d\mathbf{x} \\
 &= \int_\Omega \left[ \int_\Sigma (\nabla_s \sigma - 2\sigma H \mathbf{n}) \delta(\mathbf{x} - \mathbf{X}(\alpha, \beta, t)) dA \right] \cdot \mathbf{u}(\mathbf{x}) d\mathbf{x} \\
 &= \iint (\nabla_s \sigma - 2\sigma H \mathbf{n}) \cdot \mathbf{U}(\alpha, \beta, t) |\mathbf{X}_\alpha \times \mathbf{X}_\beta| d\alpha d\beta \quad (\text{since } \mathbf{U}(\alpha, \beta, t) = \mathbf{u}(\mathbf{X}(\alpha, \beta, t), t)) \\
 &= \iint \left[ (\sigma(\mathbf{X}_\beta \times \mathbf{n}))_\alpha + (\sigma(\mathbf{n} \times \mathbf{X}_\alpha))_\beta \right] \cdot \mathbf{U}(\alpha, \beta, t) d\alpha d\beta \quad (\text{from Lemma 1(ii)}) \\
 &= - \iint \sigma(\mathbf{X}_\beta \times \mathbf{n}) \cdot \mathbf{U}_\alpha + \sigma(\mathbf{n} \times \mathbf{X}_\alpha) \cdot \mathbf{U}_\beta d\alpha d\beta \quad (\text{by integration by parts}) \\
 &= - \iint \sigma(\nabla_s \cdot \mathbf{U}) |\mathbf{X}_\alpha \times \mathbf{X}_\beta| d\alpha d\beta \quad (\text{by Eq. (9)}) \\
 &= - \int_\Sigma \sigma(\nabla_s \cdot \mathbf{U}) dA = -\langle \sigma, \nabla_s \cdot \mathbf{U} \rangle_\Sigma. \quad \square
 \end{aligned}$$

It is important to mention that when the surface is incompressible  $\nabla_s \cdot \mathbf{U} = 0$ , the rate of change of work done by the elastic force equals to zero which has no surprise since the tension here acts like a Lagrange's multiplier to enforce the surface incompressibility constraint. However, if the bending force  $\mathbf{F}_b$  (the detailed form can be seen in [2]) is added in Eq. (3), then the rate of change of work done by the elastic membrane in Theorem 1 should become  $\frac{dW}{dt} = \langle \mathbf{S}[\mathbf{F}_\sigma] + \mathbf{S}[\mathbf{F}_b], \mathbf{u} \rangle_\Omega = -\langle \sigma, \nabla_s \cdot \mathbf{U} \rangle_\Sigma + \langle \mathbf{F}_b, \mathbf{U} \rangle_\Sigma = \langle \mathbf{F}_b, \mathbf{U} \rangle_\Sigma$  whenever  $\nabla_s \cdot \mathbf{U} = 0$ . So the elastic tension force does not do extra work to the fluid, but the bending force does.

### 3. Formulations of nearly incompressible surface

As mentioned earlier, the tension  $\sigma(\alpha, \beta, t)$  in Eq. (3) is an unknown variable and must be solved as part of solutions which complicates the numerical procedures in practice. To avoid solving the unknown tension to enforce the surface incompressibility, we adopt a nearly surface incompressible approach and introduce two different elastic penalty energies that results in different tensions. Nevertheless, those resultant elastic forces have the same mathematical form as Eq. (3) despite the fact that the associated elastic tensions are different as we show next.

**Theorem 2.** For the following elastic penalty energies  $E_1$  and  $E_2$  with the corresponding tensions  $\sigma_1$  and  $\sigma_2$ , their resultant elastic forces have the exactly same mathematical form as in Eq. (3). Here, the penalty constant  $\sigma_0$  is chosen to be sufficiently large and the superscript  $\mathbf{X}^0$  denotes the initial configuration of the surface.

(i) Relative difference penalty energy

$$E_1 = \frac{\sigma_0}{2} \iint \left( \frac{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} - 1 \right)^2 |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| d\alpha d\beta, \quad \sigma_1 = \sigma_0 \left( \frac{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} - 1 \right).$$

(ii) Logarithmic difference penalty energy

$$E_2 = \frac{\sigma_0}{2} \iint \left( \ln \left( \frac{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} \right) \right)^2 |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| d\alpha d\beta, \quad \sigma_2 = \sigma_0 \frac{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} \ln \left( \frac{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} \right).$$

**Proof.** We perturb the configuration  $\mathbf{X}$  by  $\mathbf{Y}$  so the energy becomes

$$E_1(\mathbf{X} + \varepsilon \mathbf{Y}) = \frac{\sigma_0}{2} \iint \left( \frac{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} - 1 \right)^2 |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| d\alpha d\beta,$$

where  $\varepsilon$  is a small parameter. By taking the derivative with respect to  $\varepsilon$  and setting  $\varepsilon = 0$ , we obtain

$$\begin{aligned} \frac{d}{d\varepsilon} E_1(\mathbf{X} + \varepsilon \mathbf{Y}) \Big|_{\varepsilon=0} &= \iint \sigma_0 \left( \frac{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} - 1 \right) \frac{1}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} \\ &\quad \times \frac{(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta}{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|} \cdot [\mathbf{Y}_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta + (\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times \mathbf{Y}_\beta] \Big|_{\varepsilon=0} |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| d\alpha d\beta \\ &= \iint \sigma_0 \left( \frac{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} - 1 \right) \frac{\mathbf{X}_\alpha \times \mathbf{X}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} \cdot (\mathbf{Y}_\alpha \times \mathbf{X}_\beta + \mathbf{X}_\alpha \times \mathbf{Y}_\beta) d\alpha d\beta \\ &= \iint \sigma_1 \mathbf{n} \cdot (\mathbf{Y}_\alpha \times \mathbf{X}_\beta + \mathbf{X}_\alpha \times \mathbf{Y}_\beta) d\alpha d\beta \quad (\text{by the definition of } \sigma_1 \text{ in (i)}) \\ &= \iint \sigma_1 (\mathbf{X}_\beta \times \mathbf{n}) \cdot \mathbf{Y}_\alpha + \sigma_1 (\mathbf{n} \times \mathbf{X}_\alpha) \cdot \mathbf{Y}_\beta d\alpha d\beta \quad (\text{by the scalar triple product formula}) \\ &= - \iint (\sigma_1 (\mathbf{X}_\beta \times \mathbf{n})_\alpha \cdot \mathbf{Y} + (\sigma_1 (\mathbf{n} \times \mathbf{X}_\alpha)_\beta \cdot \mathbf{Y} d\alpha d\beta \quad (\text{by integration by parts}) \\ &= - \iint (\nabla_s \sigma_1 - 2\sigma_1 H \mathbf{n}) \cdot \mathbf{Y} |\mathbf{X}_\alpha \times \mathbf{X}_\beta| d\alpha d\beta \quad (\text{from Lemma 1(ii)}) \\ &= - \int_{\Sigma(t)} (\nabla_s \sigma_1 - 2\sigma_1 H \mathbf{n}) \cdot \mathbf{Y} dA \quad (\text{since } dA = |\mathbf{X}_\alpha \times \mathbf{X}_\beta| d\alpha d\beta) \\ &= \int_{\Sigma(t)} \frac{\delta E_1}{\delta \mathbf{X}} \cdot \mathbf{Y} dA = \int_{\Sigma(t)} -\mathbf{F}_\sigma \cdot \mathbf{Y} dA. \end{aligned} \tag{10}$$

According to the principle of virtual work, this concludes that the resultant force has the same mathematical form as shown in Eq. (3).

Similarly, the perturbed energy of  $E_2$  can be written as

$$E_2(\mathbf{X} + \varepsilon \mathbf{Y}) = \frac{\sigma_0}{2} \iint \left( \ln \frac{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} \right)^2 |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| d\alpha d\beta.$$

By taking the derivative with respect to  $\varepsilon$  and setting  $\varepsilon = 0$ , it yields

$$\begin{aligned} \frac{d}{d\varepsilon} E_2(\mathbf{X} + \varepsilon \mathbf{Y}) \Big|_{\varepsilon=0} &= \iint \sigma_0 \left( \ln \frac{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} \right) \frac{1}{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|} \\ &\quad \times \frac{(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta}{|(\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta|} \cdot [\mathbf{Y}_\alpha \times (\mathbf{X} + \varepsilon \mathbf{Y})_\beta + (\mathbf{X} + \varepsilon \mathbf{Y})_\alpha \times \mathbf{Y}_\beta] \Big|_{\varepsilon=0} |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| d\alpha d\beta, \\ &= \iint \sigma_0 \frac{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} \left( \ln \frac{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}{|\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|} \right) \frac{\mathbf{X}_\alpha \times \mathbf{X}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|} \cdot (\mathbf{Y}_\alpha \times \mathbf{X}_\beta + \mathbf{X}_\alpha \times \mathbf{Y}_\beta) d\alpha d\beta \\ &= \iint \sigma_2 \mathbf{n} \cdot (\mathbf{Y}_\alpha \times \mathbf{X}_\beta + \mathbf{X}_\alpha \times \mathbf{Y}_\beta) d\alpha d\beta \quad (\text{by the definition of } \sigma_2 \text{ in (ii)}). \end{aligned}$$

The last equation has the same form as in Eq. (10) so the rest of derivation follows the same procedure before.  $\square$

We conclude this note by discussing how those penalty energies are motivated. As shown in Eq. (6), the exact surface incompressibility means that the local stretching factor does not change as time evolves; that is,  $|\mathbf{X}_\alpha \times \mathbf{X}_\beta| = |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|$  at each Lagrangian coordinate point  $(\alpha, \beta)$ . In practice, this constraint is extremely stringent so a relaxation (nearly incompressible approach) can be adopted to make  $|\mathbf{X}_\alpha \times \mathbf{X}_\beta| \approx |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|$ . Instead of finding the unknown tension, one can choose the elastic tension with sufficiently large penalty constant  $\sigma_0$  as in (i) to keep the ratio  $|\mathbf{X}_\alpha \times \mathbf{X}_\beta| / |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|$  close to one. This is exactly how the elastic energy  $E_1$  and the tension  $\sigma_1$  are derived. Alternatively, one can rewrite Eq. (6) as  $\frac{\partial}{\partial t} (\ln |\mathbf{X}_\alpha \times \mathbf{X}_\beta|) = \nabla_s \cdot \mathbf{U}$ . Similarly, one can derive the elastic energy  $E_2$  as in (ii) so the corresponding elastic tension becomes  $\sigma_2$ . Again, this nearly incompressible approach is to make logarithmic difference of the surface stretching factors such as  $\ln(|\mathbf{X}_\alpha \times \mathbf{X}_\beta| / |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0|) = \ln |\mathbf{X}_\alpha \times \mathbf{X}_\beta| - \ln |\mathbf{X}_\alpha^0 \times \mathbf{X}_\beta^0| \approx 0$ . A similar formulation of (i) is used by the present authors in [2]; however, the logarithmic difference penalty energy is new to the best of our knowledge. One should, however, mention that Hencky model is based on the logarithmic behavior of one dimensional strain and extended to three dimensions providing a comparison with the experimental results of an elastic material such as vulcanized rubber [7]. By using the elastic tension  $\sigma_1$  or  $\sigma_2$ , we can circumvent the difficulty of finding the unknown tension in vesicle simulations to reduce computational complexity. Meanwhile, both elastic tension show similar effectiveness to keep the nearly incompressibility. Although not shown here due to limited space, we have implemented the above both approaches in our 3D code and the simulation results are in good agreements.

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