SIAM J. NUMER. ANAL.
Vol. 38, No. 6, pp. 1902–1925

A METHOD FOR COMPUTING NEARLY SINGULAR INTEGRALS*

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Abstract. We develop a method for computing a nearly singular integral, such as a double layer potential due to sources on a curve in the plane, evaluated at a point near the curve. The approach is to regularize the singularity and obtain a preliminary value from a standard quadrature rule. Then we add corrections for the errors due to smoothing and discretization, which are found by asymptotic analysis. We prove an error estimate for the corrected value, uniform with respect to the point of evaluation. One application is a simple method for solving the Dirichlet problem for Laplace's equation on a grid covering an irregular region in the plane, similar to an earlier method of A. Mayo [*SIAM J. Sci. Statist. Comput.*, 6 (1985), pp. 144–157]. This approach could also be used to compute the pressure gradient due to a force on a moving boundary in an incompressible fluid. Computational examples are given for the double layer potential and for the Dirichlet problem.

Key words. nearly singular integrals, potential theory, Dirichlet problem, fluid interfaces

AMS subject classifications. 65D30, 65N99, 31A10, 35J25

PII. S0036142999362845

1. Introduction.

The solutions to many problems in differential equations can be expressed in terms of singular integrals. Perhaps the most familiar example is the representation of solutions of boundary value problems for Laplace's equation or Poisson's equation. The computation of such integrals often requires special methods. It seems that the most difficult case to compute accurately is the *nearly singular* integral which occurs when the point of evaluation is close to a singular source point but not identical with it. For example, for a double layer potential on a curve, the integrand is smooth when the point is on the curve but has large derivatives when the point of evaluation is off the curve but close by. The values of such integrals might be needed if the curve is embedded in a region covered by a grid, and we wish to compute values of the potential at grid points near the curve. If we calculate the integral at one point with a standard quadrature rule, the approximation is high order accurate; however, for fixed grid size, the error is typically much larger at points near the curve than for points further away, and high resolution or a special method is needed to obtain accurate values at these nearby points (cf. [1, section 7.2.1]). For example, if we calculate the double layer potential by the trapezoidal rule with singularity subtraction (see (1.5), (1.7)), the accuracy, uniform with respect to location, is only first order. That is, with grid spacing h on the curve, the error is bounded by Ch for some C, uniformly in a neighborhood of the curve, and the power of h in this statement cannot be improved; this is explained and illustrated below.

In this paper we develop a simple and efficient method for computing nearly singular integrals. While the method is very general, we treat here the specific case of a double layer potential on a curve in two dimensions (see (1.1)), or the gradient

*Received by the editors October 18, 1999; accepted for publication (in revised form) August 24, 2000; published electronically January 19, 2001.

http://www.siam.org/journals/sinum/38-6/36284.html

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of a single or double layer potential. The approach is this: We replace the singular kernel of the integral with a regularized or smoothed version. We approximate the integral using a standard quadrature. The value so obtained may have significant errors because of the smoothing and the discretization of an integrand which has large derivatives. We now compute corrections to the preliminary quadrature, for each of the two types of errors, based on asymptotic analysis of the integral near the point of evaluation. With the corrections presented here, the computed value of the double layer potential has an error of two parts, a smoothing error of $O(\delta^3)$ and a discretization error of $O(h^2)$; here δ is the radius of smoothing, h is the grid spacing in the parameter describing the curve, and we assume that δ/h is constant. In many cases the smoothing error $O(\delta^3)$ is dominant, and we typically observe a combined error of $O(h^3)$ in our numerical examples. (Higher order corrections could be computed by the same method presented here, but they would be more complicated.) The error is uniform with respect to the location, and the work does not increase as the point of evaluation approaches the curve. One important limitation is that the curve is required to be smooth; i.e., this method would not be valid for a curve with corners without further modification. For the double layer potential on a curve, the approach works even without regularization; after the discretization correction, it results in an error of $O(h^2)$. We expect, however, that the regularization is essential for this method in cases where the integrand is unbounded, as for potentials on surfaces in three-space.

The evaluation of the integral requires information about the curve and integrand only at regularly spaced points; no special quadrature points or subdivisions are needed. This could be an advantage in a context where limited information is available, as would be the case when a moving curve is computed. For greatest efficiency, this method could be used in conjunction with a rapid summation method such as the fast multipole method (FMM) [13] to obtain values of the potential at a set of points near the curve. Suppose we use $J = O(h^{-1})$ quadrature points on the curve, with spacing $O(h)$, and we wish to evaluate the potential at M grid points in the plane with grid size $O(h)$. The FMM could be used to calculate the part of the summation corresponding to pairs of points which are separated by distance $O(h)$ for which the regularization is insignificant; this requires $O(M + J)$ operations with a constant depending on a specified error tolerance. The remaining $O(M + J)$ pairs could be calculated directly. The correction terms require $O(M)$ operations. Thus, the total operation count is $O(M+J)$. For example, if the M points are those within a few grid spacings of the curve, then $M = O(h^{-1})$, and the operation count is $O(h^{-1})$. In this way values can be obtained rapidly for a large set of points, without special care, and this fact should be of practical advantage.

As an application we present in section 4 a method for solving boundary value problems for Laplace's equation in irregular regions. It is very similar to the method introduced by Mayo [22] but more direct. To solve the Dirichlet problem inside a curve, we write the solution as a double layer potential, solving an integral equation on the curve for the dipole moment. We introduce a grid on a rectangle containing the region of interest. We compute values of the solution from the double layer potential at grid points near the curve in the manner described above. From these values we can form the discrete Laplacian of the solution at points where the stencil crosses the curve. Setting the Laplacian to zero at other grid points, we then invert, using a fast solver for the discrete Laplacian on the rectangular grid. Thus we obtain the values of the solution at all grid points. In our experiments we have used both the five-point

and the nine-point Laplacian. We find convergence to second order or better, with improved accuracy using the nine-point Laplacian. The simplicity of this approach could be an advantage if, e.g., such an elliptic problem occurs at each time step of a time-dependent calculation. As noted by Mayo, another advantage is that exterior problems are as easy to deal with as interior ones; no artificial boundary conditions are needed at computational boundaries. (Of course, the solution could also be found at all grid points by computing the integrals directly; the FMM could again be used.)

This approach could be used to compute a pressure gradient in two-dimensional fluid flow due to forces concentrated on a curve. Such a pressure gradient is often needed when the Navier–Stokes equations of viscous, incompressible flow are solved numerically and a force is exerted by a boundary within the fluid. The force might be the surface tension on a boundary separating two fluids (cf. [6]), or an elastic force on a membrane (cf. [26, 27]). As explained in section 5, finding the corresponding pressure gradient amounts to computing the gradient of a single or double layer potential due to sources on the curve. The discrete Laplacian can be used again to find the values on the entire grid from those at points near the curve. If viscous flow is computed by the projection method, for instance, a preliminary velocity field is projected onto the space of divergence-free vector fields. This projection is expressed by subtracting a pressure gradient [26, 27].

The method developed here generalizes that of [3] for computing singular integrals on $\mathbf{R}^d$ or layer potentials on doubly periodic surfaces in $\mathbf{R}^3$. In that case the integral was evaluated at grid points on the surface. Here we restrict attention to integrals on curves, but the evaluation of the integral at an arbitrary point requires a more general point of view. Some existing methods for computing nearly singular integrals can be found in [17, 14, 28]. Further use of Mayo's methods for elliptic problems can be found in [21, 23, 24, 25] as well as in [22].

We now describe the computation of a double layer potential due to sources on a curve bounding a region in the plane. Let $\Omega \subseteq \mathbf{R}^2$ be a bounded region with boundary curve $\mathcal{C}$. We suppose that $\mathcal{C}$ is known in parametrized form $\mathcal{C} = \{x(\alpha) \in \mathbf{R}^2 : 0 \leq \alpha \leq 2\pi\}$, going counterclockwise, and that $\alpha \mapsto x(\alpha)$ is the restriction of a smooth, periodic function. A double layer potential on $\mathcal{C}$ has the form

$$u(y) \;=\; \int_{\mathcal{C}} \frac{\partial G}{\partial n_x}(x-y) F(x)\, ds(x), \tag{1.1}$$

where n_x is the unit outward normal at $x \in \mathcal{C}$, s is arclength, and F is the dipole moment. $G(x)$ is the fundamental solution for the Laplacian in $\mathbf{R}^2$, $G(x) = (2\pi)^{-1} \log|x|$, so that $\Delta G(x) = \delta(x)$, and

$$\frac{\partial G}{\partial n_x}(x-y) \;=\; n_x \cdot \nabla G(x-y) \;=\; \frac{1}{2\pi} \frac{n_x \cdot (x-y)}{|x-y|^2}\,. \tag{1.2}$$

As shown in potential theory (e.g., see [7]), $\Delta u = 0$ on $\mathbf{R}^2 - \mathcal{C}$, and u has different limiting values on $\mathcal{C}$, when approached from the interior region Ω and the exterior region $\mathbf{R}^2 - \bar{\Omega}$. The integrand is smooth for $y \in \mathcal{C}$, but for y close to $\mathcal{C}$ it is nearly singular. In parametrized form (1.1) is

$$u(y) \;=\; \int_0^{2\pi} \frac{\partial G}{\partial n(\alpha)}(x(\alpha)-y) f(\alpha)\, ds(\alpha) \;=\; \int_0^{2\pi} N(\alpha) \cdot \nabla G(x(\alpha)-y) f(\alpha)\, d\alpha. \tag{1.3}$$

Here $f(\alpha) = F(x(\alpha))$, $n(\alpha) = n_{x(\alpha)}$, and $N(\alpha) = (x_2'(\alpha), -x_1'(\alpha))$, so that $n(\alpha) = N(\alpha)/|N(\alpha)|$.

Now suppose we want to calculate u at a point y near $\mathcal{C}$. Then, y is along some normal line through $\mathcal{C}$, i.e., $y = x(\alpha_0) + bn(\alpha_0)$ for some α_0 and some $b \in \mathbf{R}$. First we rewrite the integral to reduce the order of singularity, using the identity

$$\int_{\mathcal{C}} \frac{\partial G}{\partial n_x}(x-y)\, ds(x) \;=\; \begin{cases} 1, & y \in \Omega, \\ 0, & y \in \mathbf{R}^2 - \bar{\Omega}. \end{cases} \tag{1.4}$$

Assuming we can find α_0 and b, given y, we write

$$u(y) \;=\; \int_0^{2\pi} N(\alpha)\cdot\nabla G(x(\alpha)-y)\,[f(\alpha)-f(\alpha_0)]\; d\alpha \;+\; \chi(y) f(\alpha_0), \tag{1.5}$$

where $\chi(y) = 1$ for $y \in \Omega$, $\chi(y) = 0$ for $y \in \mathbf{R}^2 - \bar{\Omega}$. Both formulas extend continuously to $\mathcal{C}$.

Next, before discretizing the integral, we replace ∇G with a regularized version

$$\begin{aligned} \nabla G_\delta(x(\alpha)-y) \;&=\; (1-e^{-r^2/\delta^2})\nabla G(x(\alpha)-y) \\ &=\; (2\pi r^2)^{-1}(1-e^{-r^2/\delta^2})(x(\alpha)-y)\,, \qquad r = |x(\alpha)-y|, \end{aligned} \tag{1.6}$$

where δ is a smoothing parameter to be chosen. (∇G_δ is the gradient of the indefinite integral G_δ of $(2\pi r)^{-1}(1-e^{-r^2/\delta^2})$. G_δ can be thought of as the convolution of G with the approximate delta function $(\pi\delta^2)^{-1}e^{-r^2/\delta^2}$; see [4].) Now let S be the trapezoidal sum for the integral in (1.5) with grid points $\alpha_j = jh$, $1 \le j \le J$, $h = 2\pi/J$:

$$S \;=\; \sum_{j=1}^{J} N(\alpha_j)\cdot\nabla G_\delta(x(\alpha_j)-y)\,[f(\alpha_j)-f(\alpha_0)]\; h. \tag{1.7}$$

If we view S as an approximation to the integral in (1.5), there are two errors: the smoothing error from replacing ∇G by ∇G_δ in the integral, and the quadrature error from replacing the integral, with ∇G_δ, by the sum S. Symbolically,

$$\Sigma_\delta - \int \;=\; \left(\int_\delta - \int\right) + \left(\Sigma_\delta - \int_\delta\right). \tag{1.8}$$

The theory of sections 2 and 3 shows that these errors are $O(\delta^2)$ and $O(h)$, respectively, assuming for the latter that $\delta = O(h)$ as $h \to 0$. Moreover, corrections are derived so that the errors can be improved to $O(\delta^3)$ and $O(h^2)$. The correction for the smoothing is

$$T_1 \;=\; -\,\delta^2(4\pi)^{-1}\eta\left(\sqrt{\pi}e^{-\eta^2} - \pi|\eta|\,\mathrm{erfc}\,|\eta|\right)\left(\tau_0^{-2}f_0'' - \tau_0^{-4}(x_0''\cdot x_0')f_0'\right). \tag{1.9}$$

Here $\eta = b/\delta$; $x_0' = x'(\alpha_0)$, $f_0' = f'(\alpha_0)$, etc; $\tau_0 = s'(\alpha_0) = |x'(\alpha_0)|$; and erfc is the complementary error function

$$\mathrm{erfc}(z) \;=\; \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\zeta^2}\, d\zeta\,. \tag{1.10}$$

The correction for the quadrature is

$$T_2 \;=\; -\,\frac{hf_0'\eta\sigma}{2}\sum_{n=1}^{\infty}\sin(2n\pi\alpha_0/h)E(\eta,n\pi\sigma), \tag{1.11}$$

where $\sigma = \delta/h\tau_0$ and

$$E(\eta,\zeta) \;=\; e^{2\eta\zeta}\operatorname{erfc}(\eta+\zeta) + e^{-2\eta\zeta}\operatorname{erfc}(-\eta+\zeta)\,. \tag{1.12}$$

We summarize the conclusions in the following statement. We assume throughout that the curve $\mathcal{C}$ and the integrand f are C^∞; in fact, the error estimates depend on a few derivatives of each.

Main Theorem. *Let $\tilde u(y)$ be the approximation to $u(y)$ in* (1.3) *or* (1.5)*, computed as*

$$\tilde u(y) \;=\; S + T_1 + T_2 + \chi f(\alpha_0) \tag{1.13}$$

with S, T_1, T_2 as in (1.7), (1.9)–(1.12)*. Then, assuming $\delta = \rho h$ with ρ constant as $h \to 0$,*

$$\tilde u(y) - u(y) \;=\; \varepsilon_1 + \varepsilon_2, \tag{1.14}$$

where $|\varepsilon_1| \le C_1\delta^3$ and $|\varepsilon_2| \le C_2h^2$, uniformly for y near $\mathcal{C}$. Here ε_1 is the smoothing error and ε_2 is the quadrature error.

This theorem follows from Theorems 2.1 and 3.2. There are several remarks to be made. We have not combined the two errors in (1.14) because in some circumstances ε_2 can be kept small so that ε_1 is the dominant error; this is done by ensuring that $\sigma = \delta/h\tau_0$ is not too small (see below). The error bounds are inequalities, not asymptotic equalities for fixed y. Thus, it does not seem practical to use extrapolation to improve the order. Note, e.g., that for fixed α_0, the correction T_1 is a function of $(y - x(\alpha_0))/\delta$. The theory of section 2 suggests that the same is true for ε_1.

The sum in T_2 is infinite, but the terms decay very rapidly, so that only a few terms need to be computed. Specifically, $E(\eta,\zeta) \le 3e^{-\zeta^2}$, showing that the terms $E(\eta, n\pi\sigma)$ have Gaussian decay in n. Note that the spacing of points on the curve is about $h\tau_0$ near $\alpha = \alpha_0$, so that σ measures the radius of smoothing relative to the point spacing. The quadrature correction T_2 goes to zero rapidly as σ increases. For example, if $\sigma \ge 2$, it can be checked that $|T_2| \le 3h|f_0'||\eta| \cdot 10^{-17}$. Moreover, $E(\eta,\zeta) \le 3e^{-2\eta\zeta}$, and thus T_2 decays rapidly as $\eta = b/\delta$ increases, as we should expect, since the integrand is smooth for y away from $\mathcal{C}$. For example, if $\sigma \ge 2$ and $\eta \ge 3$, $|T_2| \le 4h|f_0'| \cdot 10^{-16}$. On the other hand, making δ large will likely increase the smoothing error ε_1. In the sine factor in T_2, we could replace α_0/h by the remainder ν when α_0 is divided by h, i.e., $\alpha_0 = \ell h + \nu h$ for some integer ℓ; cf. section 3. We might choose our mesh on the curve as $\alpha_j = (j + 1/2)h$ rather than jh; it can be seen that this has the effect of reversing the sign of T_2.

The double layer potential can be computed without regularization, by a similar method, and the results are qualitatively similar. That is, we can compute the sum S as in (1.7), but with ∇G rather than ∇G_δ. The quadrature error is again $O(h)$. A correction corresponding to T_2 can be calculated, either by taking the formal limit of (1.11) as $\delta \to 0$, or by a calculation similar to that of section 3. It is

$$T_2 \;=\; -\frac{f_0'b}{\tau_0}\sum_{n=1}^{\infty}\sin(2n\pi\nu)\exp(-2n\pi|b|/h\tau_0) \;=\; -\frac{f_0'b}{2\tau_0}\,\frac{\sin 2\pi\nu}{\cosh(2\pi|b|/h\tau_0) - \cos 2\pi\nu}. \tag{1.15}$$

(It can be checked that T_2/h is a bounded function.) Now we can form the corrected approximation $\tilde{u}(y)$ to $u(y)$ as in (1.13), omitting T_1. Then, as in the Main Theorem, $\tilde{u}(y) - u(y) = \varepsilon_2$, with $|\varepsilon_2| \le C_2 h^2$, uniformly in y. This can be shown by slight modification of the analysis of section 3.

Example. We choose $\mathcal{C}$ to be the unit circle, parametrized as $(\cos\alpha, \sin\alpha)$, and $f(\alpha) = 2\sin 3\alpha$. Choosing h so that $J = 2\pi/h$ is a multiple of 25, we take $\alpha_0 = 2\pi/25 + h/10$ and $y = (1 + h/10)x(\alpha_0)$; thus y is distance $h/10$ from the curve and displaced along the curve by $h/10$ from the closest grid point. Computing the sum S as in (1.7), with no regularization, for $J = 2\pi/h = 50 \cdot 2^k$, for $0 \le k \le 4$, we find the error is about $.32\,h$. After adding the correction T_2 of (1.15), the error in $\tilde{u}(y)$ is about $.020\,h^2$. On the other hand, if we compute $\tilde{u}(y)$ as in (1.13), using the regularized method, with $\sigma = \delta/h = 1$, we find the error is about $.45\,h^3$; with $\delta/h = 2$, it is $3.5\,h^3$.

Finally, we outline the contents of the remaining sections. In section 2 we derive the smoothing correction T_1 by asymptotic analysis of the integral near $\alpha = \alpha_0$. In section 3 we compute the quadrature correction T_2, after first deriving a general result for quadrature of a nearly singular integral over a hyperplane with homogeneous kernel. The analysis here generalizes that in [3, section 3]. In section 4 we explain the application to the Dirichlet problem. We also give related quadrature formulas for an integral like (1.3) with tangential gradient of G and show how the two cases can be used to find the gradient of a single or double layer potential. In section 5 we discuss the possible application to computing pressure gradients in fluids. Computational examples are presented in section 6 which demonstrate the performance of the method for computing singular integrals near the curve and of the method for solving the Dirichlet problem.

2. The smoothing error. In this section we find the largest contribution to the error in the double layer potential which results from replacing the fundamental solution G by the regularized version G_δ. In other contexts the smoothing error in a regularized kernel has been made higher order by imposing moment conditions on the choice of the kernel [3, 4, 15], but that does not appear to be possible for the nearly singular case. We first state the conclusion; for simplicity we assume $\alpha_0 = 0$, $x(0) = 0$, and $f(0) = 0$.

THEOREM 2.1. *With the curve $\mathcal{C}$ as before, assume that $x(0) = 0$ and y is along the normal line through $x = 0$, so that $y = bn_0$ for some b. Also assume $f(0) = 0$. Let ε be the error due to the smoothing*

$$\varepsilon = \int_0^{2\pi} N(\alpha) \cdot \nabla G_\delta(x(\alpha) - y) f(\alpha)\, d\alpha - \int_0^{2\pi} N(\alpha) \cdot \nabla G(x(\alpha) - y) f(\alpha)\, d\alpha\,. \tag{2.1}$$

Then, $\varepsilon = -T_1 + O(\delta^3)$ as $\delta \to 0$, where T_1 is given by (1.9). *The error estimate is uniform in y near $\mathcal{C}$ but depends on derivatives of $\mathcal{C}$ and f.*

Proof. Writing out ∇G_δ and ∇G, we have

$$\varepsilon = -\frac{1}{2\pi} \int \frac{n(\alpha) \cdot (x(\alpha) - y)}{r^2} e^{-r^2/\delta^2} f(\alpha)\, ds(\alpha) \tag{2.2}$$

with $r = |x(\alpha) - y|$. For small δ, the error is very small when r is at least $O(1)$. Thus, we may assume that f is zero outside a neighborhood $\mathcal{N}$ of $x = 0$, determined by the geometry, and that $y \in \mathcal{N}$. We can use any parametrization in the integral (2.1), and for simplicity we will assume that the parameter α is the arclength s. At the end we

will modify the result for arbitrary α. Thus, we write (2.1) as

$$\varepsilon = -\frac{1}{2\pi}\int \frac{n(s)\cdot(x(s)-y)}{r^2} e^{-r^2/\delta^2} f(s)\,ds. \tag{2.3}$$

To begin we approximate $r = |x(s)-y| = |x(s)-bn_0|$ for s, b near 0, using the Taylor expansion of $x(s)$. With $' = d/ds$, $x_0' = x'(0)$, etc., we have $x(s) = x_0' s + \frac{1}{2}x_0'' s^2 + O(s^3)$. Since s is arclength, $|x'(s)| = 1$, and the unit normal is $n(s) = x'(s)^\perp = n_0 + (x_0'')^\perp s + O(s^2)$, where $v^\perp$ means $(v_2, -v_1)$ for a vector $v = (v_1, v_2)$. Also $x_0''\cdot x_0' = 0$, so that $x_0'' = \kappa n_0$, $(x_0'')^\perp = -\kappa x_0'$, and

$$x(s) - y = x_0' s + (\tfrac{1}{2}\kappa s^2 - b)n_0 + O(s^3), \tag{2.4}$$

$$n(s) = n_0 - \kappa x_0' s + O(s^2). \tag{2.5}$$

Then,

$$r^2 = |x-y|^2 = s^2 + b^2 - \kappa b s^2 + O(s^4) + O(bs^3). \tag{2.6}$$

To simplify the dependence of r, we now define a new parameter ξ on the curve so that $r^2 = \xi^2 + b^2$, i.e.,

$$\xi^2 = (1-\kappa b)s^2 + O(s^4) + O(bs^3) \tag{2.7}$$

or

$$\xi = (1 - \tfrac{1}{2}\kappa b)s + O(s^3) + O(b^2 s). \tag{2.8}$$

Assuming that $\mathcal{N}$ is small enough, we can solve for $s = s(\xi, b)$, obtaining

$$s = (1 + \tfrac{1}{2}\kappa b)\xi + O(\xi^3) + O(b^3). \tag{2.9}$$

Now we can write the error ε as

$$\varepsilon = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{n(s)\cdot(x(s)-y)}{\xi^2+b^2} e^{-(\xi^2+b^2)/\delta^2} f(s)\frac{ds}{d\xi}\,d\xi. \tag{2.10}$$

When δ is small, the largest contribution will come from the lowest powers of (ξ, b) in the Taylor expansion of the factor multiplying the radial part. To see this, we note that for a typical term $\xi^m b^n$, with $m+n \geq 2$, we have

$$\int_{-\infty}^{\infty} \frac{e^{-(\xi^2+b^2)/\delta^2}}{\xi^2+b^2}\xi^m b^n\,d\xi = \delta^{m+n-1} M(b/\delta), \tag{2.11}$$

where $M(b/\delta)$ is uniformly bounded, as we find by rescaling the variables to $\xi/\delta, b/\delta$. Thus, we will expand the integrand above in s and convert to ξ. Using (2.4), (2.5), (2.9) we get

$$n\cdot(x-y) = -b - \tfrac{1}{2}\kappa s^2 + O(s^3) + O(b^3) = -b - \tfrac{1}{2}\kappa\xi^2 + O(\xi^3) + O(b^3) \tag{2.12}$$

and similarly

$$f(s) = f_0' s + \tfrac{1}{2}f_0'' s^2 + O(s^3) = f_0'\xi + \tfrac{1}{2}f_0'\kappa b\xi + \tfrac{1}{2}f_0''\xi^2 + O(\xi^3) + O(b^3), \tag{2.13}$$

$$\frac{ds}{d\xi} = 1 + \tfrac{1}{2}\kappa b + O(\xi^2) + O(b^2). \tag{2.14}$$

Multiplying these, we have

$$\varepsilon = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-(\xi^2+b^2)/\delta^2}}{\xi^2+b^2} w(\xi,b)\,d\xi, \tag{2.15}$$

where

$$\begin{aligned} w(\xi,b) &= -(n\cdot(x-y))\,f(s)(ds/d\xi) \\ &= f_0'\xi b + \tfrac{1}{2}f_0'\kappa\xi^3 + f_0'\kappa\xi b^2 + \tfrac{1}{2}f_0''\xi^2 b + R(\xi,b)\,, \quad R = O(\xi^4+b^4). \end{aligned} \tag{2.16}$$

Of the five terms in (2.16), the first three are odd in ξ and thus make zero contribution to the error ε in (2.15). We discuss the remainder R next. It is a sum of terms of the form $p_4(\xi,b)q(\xi,b)$, where p_4 is a homogeneous polynomial and q is some bounded function. If we set $\xi = \delta\zeta$ and $b = \delta\eta$ in the integral, the contribution to ε from this term becomes

$$\varepsilon = \delta^{-2+4+1}\int_{-\infty}^{\infty} \frac{e^{-(\zeta^2+\eta^2)}}{\zeta^2+\eta^2} p_4(\zeta,\eta)q(\zeta/\delta,\eta/\delta)\,d\zeta. \tag{2.17}$$

The integral is bounded uniformly in η and δ since q is bounded, and thus the part of the error resulting from the remainder is $O(\delta^3)$. We are now left with the contribution from the $\xi^2 b$-term in (2.16). With the same change of variables, (2.15) is now reduced to

$$\varepsilon = \delta^2(4\pi)^{-1}\eta f_0''\int_{-\infty}^{\infty} \frac{e^{-(\zeta^2+\eta^2)}}{\zeta^2+\eta^2}\zeta^2\,d\zeta \;+\; O(\delta^3), \tag{2.18}$$

or after evaluating the integral (cf. [12, formula 3.466])

$$\varepsilon = \delta^2(4\pi)^{-1}\eta f_0''\left(\sqrt{\pi}e^{-\eta^2} - \pi|\eta|\operatorname{erfc}|\eta|\right) \;+\; O(\delta^3). \tag{2.19}$$

To finish the argument, we recall that f_0'' means the second derivative of f with respect to arclength. For an arbitrary parameter α, this can be written as $\tau_0^{-2}f_0'' - \tau_0^{-4}(x_0''\cdot x_0')f_0'$, where now $' = d/d\alpha$ and $\tau_0 = s'(0)$. Thus, the leading term in (2.19) is $-T_1$. □

3. The quadrature error. The coefficient for the quadrature correction will be derived using the Poisson summation formula: for a smooth, rapidly decreasing function f on $\mathbf{R}^d$,

$$(2\pi)^{-d/2}\sum_{j\in Z^d} f(jh)\,h^d = \sum_{n\in Z^d}\hat{f}(2\pi n/h), \tag{3.1}$$

where we write the Fourier transform as

$$\hat{f}(k) \;=\; (2\pi)^{-d/2}\int f(x)e^{-ikx}\,dx\,.$$

Another general fact which will be needed is a quadrature estimate for a mildly regular function on $\mathbf{R}^d$:

$$\left|\sum_{n\in Z^d} F(nh)\,h^d - \int_{\mathbf{R}^d} F(x)\,dx\right| \;\le\; C_\ell h^\ell \sum_{|\gamma|=\ell} |D_x^\gamma F|_{L^1}\,. \tag{3.2}$$

Here $\ell \geq d+1$, γ is a multi-index, $|\gamma| = \sum_{j=1}^{d} \gamma_j$, and C_ℓ is a universal constant. This can be proved directly from the Poisson formula; see [2, Lemma 2.2].

We begin with a general lemma which estimates the quadrature error for an integral over $\mathbf{R}^d$ with a nearly singular kernel, regularized on the scale of the grid size. We assume the kernel K is a homogeneous function of $(x, y) \in \mathbf{R}^d \times \mathbf{R}$, with some degree m:

$$K(ax, ay) = a^m K(x, y)\,, \qquad a > 0\,, \quad (x, y) \neq 0\,. \tag{3.3}$$

The regularized kernel will have the form

$$K_h(x, y) = K(x, y) s(x/h, y/h)\,, \qquad x \in \mathbf{R}^d, y \in \mathbf{R}, \tag{3.4}$$

where s is chosen so that $s \to 1$ rapidly as its argument goes to infinity.

It follows that $K_h(x, y) = h^m K_1(x/h, y/h)$. We show that the error from quadrature with the trapezoidal rule has an expansion in powers of h, and the leading term can be computed explicitly. The grid points may be shifted away from the singularity. The proof generalizes those of Lemmas 3.2–3.4 in [3], which in turn were inspired by the proof of Lemma 1 in [11] for unregularized singular integrals; expansions for errors in the latter case were derived earlier by Lyness [20]. It can be checked that the present lemma holds without regularization (i.e., with the factor of s omitted), provided $m \geq 0$ and we require $y \neq 0$.

LEMMA 3.1. *Let K_h be a smooth function on $\mathbf{R}^{d+1}$ with the form (3.4), where K and s are smooth for $(x, y) \neq 0$; K is homogeneous of degree m; $s(x, y) \to 1$ as $(x, y) \to \infty$; and $|D^\gamma s(x, y)| \leq C|(x, y)|^{-|\gamma|}$ for $|(x, y)| \geq 1$. Let f be a smooth function on $\mathbf{R}^d$ such that f and its derivatives are rapidly decreasing. We approximate the integral*

$$I = \int_{\mathbf{R}^d} K_h(x, y) f(x)\, dx = \int_{\mathbf{R}^d} K(x, y) s(x/h, y/h) f(x)\, dx \tag{3.5}$$

by the sum

$$S = \sum_{n \in \mathbf{Z}^d} K_h(nh - \nu h, y) f(nh - \nu h)\, h^d \tag{3.6}$$

with some $\nu \in \mathbf{R}^d$. Then, with $y = \eta h$,

$$S - I = h^{d+m} \left(c_0 f(0) + C_1 h + C_2 h^2 + \cdots + C_\ell h^\ell + O(h^{\ell+1}) \right), \tag{3.7}$$

where the C_j depend on η, ν, and f, for $j \geq 1$, and

$$c_0 = (2\pi)^{d/2} \sum_{n \neq 0} e^{-2\pi i n \nu} \hat{K}_1(2\pi n, \eta). \tag{3.8}$$

Here $\hat{K}_1(\cdot, \eta)$ is the Fourier transform of $K_1(\cdot, \eta)$, the sum is over $n \in \mathbf{Z}^d$, and ℓ is large depending on the smoothness of K, s, and f. The C_j and the error term are uniformly bounded with respect to $y = \eta h$ and ν on a bounded set, and c_0 is uniformly bounded with respect to η and ν.

Proof. We omit some details which are similar to those in [3]. First we take f to be a cut-off function; i.e., we assume that f has compact support and $f \equiv 1$ in

a neighborhood of $x = 0$. Using the homogeneity of K, we can write, with $x = \xi h$, $y = \eta h$,

$$\mathcal{I} \equiv h^{-(d+m)} I = \int_{\mathbf{R}^d} K(\xi, \eta) s(\xi, \eta) f(\xi h)\, d\xi \tag{3.9}$$

and

$$\mathcal{S} \equiv h^{-(d+m)} S = \sum\nolimits_n K(n - \nu, \eta) s(n - \nu, \eta) f(nh - \nu h)\,. \tag{3.10}$$

Now, using an idea from [11], we differentiate in h, keeping η fixed, and find, after returning to (x, y) variables,

$$\mathcal{I}'(h) = h^{-(d+m+1)} \int_{\mathbf{R}^d} K(x, y) s(x/h, y/h)\, x \cdot \nabla f(x)\, dx \tag{3.11}$$

and, with $\tilde{n} = n - \nu$,

$$\mathcal{S}'(h) = h^{-(d+m+1)} \sum_n K(\tilde{n}h, y) s(\tilde{n}h/h, y/h)\, \tilde{n}h \cdot \nabla f(\tilde{n}h)\, h^d. \tag{3.12}$$

We have a trapezoidal sum for a function which is zero for x outside an annular region $0 < c_1 < |x| < c_2$ with the support of ∇f; the singularity is removed, and the shift ν is immaterial. We can estimate the difference using (3.2); we note that by hypothesis

$$|D_x^k (s(x/h, y/h))| \le C|x|^{-|k|}\,, \qquad |x| \ge h, \tag{3.13}$$

and thus the derivatives of the integrand in (3.11) are bounded uniformly for $0 < h \le h_0$ and y in a bounded set. It follows from (3.2) that $|\mathcal{S}'(h) - \mathcal{I}'(h)| \le C_\ell h^{\ell-d-m-1}$ for $h > 0$ and ℓ suitably large. We may integrate from 0 to h, with $\eta = y/h$ fixed, and write $(\mathcal{S} - \mathcal{I})(h) = c_0 + O(h^{\ell-d-m})$, where c_0 is the limiting value of $\mathcal{S} - \mathcal{I}$ at $h = 0$. We conclude that $S - I = c_0 h^{d+m} + O(h^\ell)$, where the error term is uniformly bounded and c_0 depends on η and ν. In fact, c_0 depends boundedly on η and ν: for $|\eta| \ge 1$ we can choose h boundedly so that $y = \eta h = O(1)$, and we are away from the singularity; then the integrand in (3.5) has bounded derivatives, so that $h^{-(d+m)}(S(h) - I(h))$ is bounded for such h, and it follows that c_0 is bounded. Otherwise if $|\eta| \le 1$, we can choose $h = O(1)$; then $\mathcal{S}(h)$ and $\mathcal{I}(h)$ are separately bounded, since Ks is smooth, and again it follows that c_0 is bounded.

Next we extend the result to general functions f. We write f as a Taylor expansion about $x = 0$ to order ℓ with remainder and multiply by a cut-off function; the error away from the singularity is high order. The jth term in f is homogeneous in x of degree j, and the corresponding term in the integral (3.5) is similar to the case already treated, with m replaced by $m + j$. Adding these terms for $0 \le j \le \ell$ gives a sum as in (3.7). The remainder in f consists of terms of the form $q(x)g(x)$, where q is homogeneous of degree $\ell + 1$. The resulting integrand in (3.5) is $F(x, y) = K(x, y)s(x/h, y/h)q(x)g(x)$. Using the homogeneity of Kq and (3.13) we can check that $|D_x^\gamma F|_{L^1} \le C h^{m+d+\ell+1-|\gamma|}$ (cf. [3, Lemma 3.3]). Then it follows from (3.2) that the quadrature error from F is $O(h^{m+d+\ell+1})$. We have now verified (3.7); the uniform estimates follow from the derivation.

It remains to identify the constant c_0. Since (3.7) holds for arbitrary f, we assume $f(0) = 1$ and write, using (3.3),

$$c_0 = \lim_{h\to 0} h^{-(m+d)}(S - I) = \lim_{h\to 0} \left(\sum K_1(n - \nu, \eta) f(nh) - \int K_1(\xi - \nu, \eta) f(\xi h)\, d\xi \right). \tag{3.14}$$

The f-factor cuts off the diverging sum and integral at radius $O(1/h)$. We can apply the Poisson formula (3.1) to $F_h(\cdot - \nu)$, where $F_h(x) = K_1(x,\eta)f(xh + \nu h)$. The shift by ν gives an exponential factor in the transform, and we find

$$c_0 = \lim_{h\to 0} (2\pi)^{d/2} \sum_{n\neq 0} e^{-2\pi i n\nu} \hat{F}_h(2\pi n, \eta). \tag{3.15}$$

To verify (3.8) we need to show that as $h \to 0$, $\hat{F}_h$ is replaced with $\hat{K}_1$ in the limit. The hypothesis for K_1 implies that $D_x^\gamma K_1$ decays rapidly for $|\gamma|$ large; it follows that $\hat{K}_1$ and its derivatives are continuous for $k \neq 0$ and decay rapidly for large k. The limiting argument is based on this fact; see Lemma 3.4 in [3]. □

Now we apply this general lemma to the specific case of the double layer potential on a curve.

THEOREM 3.2. *Suppose the curve $\mathcal{C}$ is parametrized by $x(\alpha)$ with $x(0) = 0$. Let $\tau = |x'(0)|$, and let n_0 the unit normal at 0, $n_0 = N(0)/\tau$. Let $y = bn_0$ be a point on the normal line through 0. Let $f(\alpha)$ be a smooth function on the curve with $f(0) = 0$. With ∇G_δ as in (1.6), we write the regularized double layer potential due to f at y as*

$$I = \int_0^{2\pi} N(\alpha) \cdot \nabla G_\delta(x(\alpha) - y) f(\alpha)\, d\alpha \tag{3.16}$$

and approximate I by the sum

$$S = \sum_{j=1}^{J} N(\alpha_j) \cdot \nabla G_\delta(x(\alpha_j) - y) f(\alpha_j)\, h, \tag{3.17}$$

where $\alpha_j = jh - \nu h$. Suppose we let $h \to 0$, with $\delta = \rho h$, for fixed ρ. Then

$$S - I = c_0 f'(0) h + O(h^2), \tag{3.18}$$

where

(3.19)

$$c_0 = \frac{b}{2h\tau} \sum_{n=1}^{\infty} \sin(2\pi n\nu) \left[e^{2\pi n b/h\tau} \operatorname{erfc}(b/\delta + \pi n\rho/\tau) + e^{-2\pi n b/h\tau} \operatorname{erfc}(-b/\delta + \pi n\rho/\tau) \right].$$

The error is uniform with respect to ν and y near $\mathcal{C}$, but it depends on derivatives of $\mathcal{C}$ and f.

Proof. Because of the analysis above, we can expect that the largest part of the quadrature error will come from the lowest order in α and y near the singularity. For this reason we compute the error for a simplified integrand, using the lemma above, and then show that the remainder contributes an error of higher order. Thus, we replace $x(\alpha)$ with its linear approximation $T\alpha$, where $T = x'(0)$, and $N(\alpha)$ with $N(0)$. We write $y = \beta N(0)$, with $\beta = b/|N(0)|$. We replace $f(\alpha)$ by α, assuming for simplicity that $f'(0) = 1$. We multiply by a cut-off function $\zeta(\alpha)$ such that $\zeta = 1$ near $\alpha = 0$ and $\zeta = 0$ outside some neighborhood of $\alpha = 0$. The integral and sum can then be extended to infinity; they become

$$I_1 = \int_{-\infty}^{\infty} N \cdot \nabla G_\delta(T\alpha - \beta N)\zeta(\alpha)\alpha\, d\alpha, \tag{3.20}$$

$$S_1 = \sum_{j\in\mathbf{Z}} N\cdot\nabla G_\delta(T\alpha_j - \beta N)\zeta(\alpha_j)\alpha_j\, h, \tag{3.21}$$

where $N = N(0)$. First we will show that $S_1 - I_1$ gives the stated $O(h)$ quadrature error.

Lemma 3.1 applies to (3.20), (3.21); here $d = 1, m = 0$, and consequently $S_1 - I_1 = c_0 h + O(h^2)$, with the coefficient c_0 given by (3.8). In the lemma x, y are replaced by α, β, and the regularized kernel K_h by $N\cdot\nabla G_\delta(T\alpha - \beta N)\,\alpha$. To find c_0 we need to derive the Fourier transform of the latter function with respect to α with $h = 1$ and $\delta = \rho h = \rho$; we set

$$K_1(\alpha;\beta) = N\cdot\nabla G_\rho(T\alpha - \beta N)\,\alpha\,. \tag{3.22}$$

We note for later that

$$A(\alpha,\beta) \equiv N\cdot\nabla G_\rho(T\alpha - \beta N) = -(\partial/\partial\beta)\left\{G_\rho(T\alpha - \beta N)\right\}. \tag{3.23}$$

We begin by finding the transform in (α,β) of $G_\rho(T\alpha - \beta N)$. The (α,β)-transform of the function G_ρ is

$$G_\rho\hat{}(k) \;=\; -(2\pi)^{-1}|k|^{-2}e^{-\rho^2|k|^2/4}, \tag{3.24}$$

where $k = (k_1, k_2) \in \mathbf{R}^2$. We can think of $G_\rho(T\alpha - \beta N)$ as the composition $(G_\rho \circ M)(\alpha,\beta)$, where $M(\alpha,\beta) = (T\alpha - N\beta)$. Then,

$$(G_\rho\circ M)\hat{}(k) = (\det M)^{-1}G_\rho\hat{}((M^*)^{-1}k). \tag{3.25}$$

However, $G_\rho\hat{}$ is radial, and $|(M^*)^{-1}k|^2 = (M^*M)^{-1}k\cdot k = \tau^{-2}|k|^2$, and $\det M = \tau^2$, where $\tau = |T| = |N|$. Thus,

$$(G_\rho\circ M)\hat{}(k) = \tau^{-2}G_\rho\hat{}(\tau^{-1}k), \tag{3.26}$$

$$A\hat{}(k) = (-(\partial/\partial\beta)G_\rho\circ M)\hat{}(k) = -ik_2\,\tau^{-2}G_\rho\hat{}(\tau^{-1}k). \tag{3.27}$$

Now the transform of $A(\alpha,\beta)$ in α alone is

$$\begin{aligned} A(\cdot,\beta)\hat{}(k_1) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} A\hat{}(k_1,k_2)e^{ik_2\beta}\,dk_2 \\ &= -(2\pi)^{-1/2}\tau^{-2}\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty} G_\rho\hat{}(\tau^{-1}k_1,\tau^{-1}k_2)e^{ik_2\beta}\,dk_2 \\ &= (2\pi)^{-3/2}\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty}\frac{1}{k_1^2+k_2^2}e^{-\rho^2k_1^2/4\tau^2}e^{-\rho^2k_2^2/4\tau^2}e^{ik_2\beta}\,dk_2. \end{aligned} \tag{3.28}$$

To evaluate this we use the definite integral for $a, b > 0$,

$$\int_{-\infty}^{\infty}\frac{e^{-b^2x^2}}{a^2+x^2}e^{-i\xi x}\,dx = \frac{\pi}{2a}e^{a^2b^2}\left[e^{a\xi}\operatorname{erfc}(\xi/2b + ab) + e^{-a\xi}\operatorname{erfc}(-\xi/2b + ab)\right]. \tag{3.29}$$

Substituting in (3.28) and noting some cancellation, we find

$$A(\cdot,\beta)\hat{}(k_1) = 2^{-5/2}\pi^{-1/2}\left[e^{k_1\beta}\operatorname{erfc}(\beta\tau/\rho + k_1\rho/2\tau) - e^{-k_1\beta}\operatorname{erfc}(-\beta\tau/\rho + k_1\rho/2\tau)\right], \tag{3.30}$$

assuming for the moment that $k_1 > 0$. Now $K_1 = \alpha A$, and the α-factor transforms to $i\partial/\partial k_1$, so that

(3.31)
$$\hat{K}_1(k_1, \beta) = i2^{-5/2}\pi^{-1/2}\beta \left[e^{k_1\beta}\operatorname{erfc}(\beta\tau/\rho + k_1\rho/2\tau) + e^{-k_1\beta}\operatorname{erfc}(-\beta\tau/\rho + k_1\rho/2\tau)\right].$$

Finally, we substitute this into (3.8), noting that $\hat{K}_1$ is odd in k_1, and combining terms with $\pm n$. Thus,

$$c_0 = (2\pi)^{1/2}\sum_{n=1}^{\infty}(-2i)\sin(2\pi n\nu)\hat{K}_1(2\pi n, \beta/h), \tag{3.32}$$

and with $\beta = b/\tau$ and $(\beta/h)\tau/\rho = b/\delta$, we obtain the stated formula (3.19) for c_0. This completes the calculation of the $O(h)$ term in (3.18).

Now we estimate the quadrature error arising from the difference between the integrand in (3.16) and the one we have just treated, i.e.,

$$N(\alpha)\cdot\nabla G_\delta(x(\alpha) - y)f(\alpha) \;-\; N(0)\cdot\nabla G_\delta(T\alpha - y)\zeta(\alpha)f'(0)\,\alpha. \tag{3.33}$$

For α or y away from the singularity, the derivatives are bounded uniformly in h, and the error is higher order. For this reason we multiply f by the cut-off function $\zeta(\alpha)$; we assume it is zero outside a neighborhood of $\alpha = 0$ in which $|x(\alpha) - T\alpha| \le |T\alpha|/2$. We can write (3.33) as a sum of terms, one being (with $f'(0)$ omitted)

$$\begin{aligned} F_1 &= \zeta(\alpha)\alpha\,(N(\alpha) - N(0))\cdot\nabla G_\delta(T\alpha - y) \\ &= (2\pi)^{-1}\zeta(\alpha)\alpha\,(N(\alpha) - N(0))\cdot(T\alpha - N(0)\beta)\,r_1^{-2}(1 - e^{-r_1^2/\delta^2}), \end{aligned} \tag{3.34}$$

where $r_1^2 = |T\alpha - y|^2 = |T\alpha|^2 + |y|^2 = |T|^2(\alpha^2 + \beta^2)$. The leading terms in the scalar product are $O(\alpha^2) + O(\alpha\beta)$; we can write (3.34) as a function homogeneous of degree one in (α, β) times the last factor, times a smooth function. We can easily check that $|D_\alpha^\gamma F_1| \le C|\alpha|^{1-|\gamma|}$ for $|\alpha| \ge h$, while $|D_\alpha^\gamma F_1| \le Ch^{1-|\gamma|}$ for small α. Thus, $|D_\alpha^\gamma F_1|_{L^1} \le Ch^{2-|\gamma|}$, and it follows from (3.2) that the quadrature error for F_1 is $O(h^2)$.

We can check similarly that the entire quadrature error from (3.33) is $O(h^2)$. We discuss only the most significant term, where $x(\alpha)$ replaces $T\alpha$ in ∇G_δ:

$$F_2 = \zeta(\alpha)\alpha N(0)\cdot(\nabla G_\delta(x(\alpha) - y) - \nabla G_\delta(T\alpha - y)). \tag{3.35}$$

We can write the difference of ∇G_δ as an average of $D^2 G_\delta$, multiplied by $x(\alpha) - T\alpha$. The latter has the form α^2 times a smooth function. Combining these facts, we can obtain the same pointwise estimates for F_2 as for F_1. Again $|D_\alpha^\gamma F_2|_{L^1} = O(h^{2-|\gamma|})$, and the quadrature error is $O(h^2)$. ∎

4. The Dirichlet problem. As an application of the method of computing nearly singular integrals, we describe a procedure for solving a Dirichlet problem in a bounded, two-dimensional domain with irregular boundary. We also describe modifications of this procedure for exterior boundary value problems, or for the gradient of a single or double layer potential.

The strategy for solving the Dirichlet problem is very similar to that of Mayo [22] but simpler, using the method presented here to compute double layer potentials.

Given a bounded domain Ω with smooth boundary $\partial\Omega$, we seek a harmonic function u on Ω with prescribed value on the boundary:

$$\Delta u = 0 \quad \text{on } \Omega\,, \qquad u = g \quad \text{on } \partial\Omega. \tag{4.1}$$

We suppose Ω is embedded in a region covered by a rectangular grid, and we wish to find values of u at grid points inside Ω. (See Figure 1.) By solving an integral equation on $\partial\Omega$ and writing u as a double layer potential, we can calculate values at grid points near the boundary. Now we form the discrete Laplacian for the unknown, on a grid extending Ω, and then invert the discrete Laplacian on the extended region to obtain the solution at all grid points. Now we explain the steps in more detail.

Step 1. Solve an integral equation for the dipole strength. The solution u at a point $y \in \Omega$ can be written as the double layer potential

$$u(y) = \int_{\partial\Omega} \partial_n G(x-y) f(x)\, ds(x) \tag{4.2}$$

for some function f giving the dipole strength, where $\partial_n G$ is the normal derivative with respect to n_x. If the boundary curve is parametrized as $x(\alpha)$ and we rewrite f as a function of α, then

$$u(y) = \int_0^{2\pi} N(\alpha) \cdot \nabla G(x(\alpha) - y) f(\alpha)\, d\alpha\,. \tag{4.3}$$

The function f is the solution of the integral equation on $\partial\Omega$

$$\tfrac{1}{2} f + Kf = g, \tag{4.4}$$

where K is the operator on the boundary

$$(Kf)(\alpha) = \int_0^{2\pi} K(\alpha, \alpha') f(\alpha')\, d\alpha'\,, \qquad K(\alpha, \alpha') = N(\alpha') \cdot \nabla G(x(\alpha') - x(\alpha))\,. \tag{4.5}$$

We solve the integral equation by simple iteration. The spectrum of the operator K is contained in the interval $-\frac{1}{2} < \lambda \le \frac{1}{2}$, and consequently the iteration

$$f^{n+1} = (1-\beta) f^n - 2\beta K f^n + 2\beta g \tag{4.6}$$

converges to the exact solution for $0 < \beta < 1$ (e.g., cf. [18, section 10.5] or [8, section 5.1]). To compute f, we choose grid points $\alpha_i = ih$ on the curve $\partial\Omega$ and solve the linear system

$$\tfrac{1}{2} f_i + \sum_j K(\alpha_i, \alpha_j) f_j h = g_i \tag{4.7}$$

by the iteration (4.6), with initial guess $f^0 = 2g$. Since the double layer kernel K is smooth, the integration is accurate, and so is the solution f_i (e.g., see [18, section 12.2] or [1, section 7.2]). We take the solution to have converged when successive iterates differ in L^2 norm by a prescribed tolerance, which we chose to be 10^{-12}.

Step 2. Compute the integral for the unknown at grid points near the boundary. We can now compute the solution u accurately at any point using the representation (4.3) and the method of integration introduced here. However, we only

need to do so at a few points near the boundary in order to find the solution at all grid points inside Ω. We suppose we have a grid on some rectangular region $\mathcal{R}$ containing Ω; let $\mathcal{S}$ be the set of grid points in $\mathcal{R}$. We will assume the grid size is $O(h)$, where h is the spacing on $\partial\Omega$, as both go to zero. Let $\mathcal{I} \subseteq \mathcal{S}$ be the set of *irregular* grid points in $\mathcal{R}$, i.e., those for which Δ_h requires points on both sides of $\partial\Omega$. Also let $\mathcal{J} \subseteq \mathcal{S}$ be the larger set of points needed to find Δ_h on the set $\mathcal{I}$. We compute the solution from (4.3) only on the set $\mathcal{J} \cap \Omega$, using the formulas of section 1. In doing this for a grid point $y \in \mathcal{J} \cap \Omega$, we need to write $y = x(\alpha_0) + bn(\alpha_0)$ for some α_0 and b. Since the mapping $(\alpha_0, b) \mapsto y$ is smooth with nonzero Jacobian, we can solve for α_0, b, given y, using Newton's method. To evaluate f, f', f'' at α_0 we use four-point Lagrange interpolation from the values of f at $\alpha_i = ih$.

Step 3. Form the discrete Laplacian of the extended unknown. Let U be the grid function on $\mathcal{S}$ which is the exact solution u in Ω and 0 otherwise. Also let $\tilde{U}$ be the grid function on $\mathcal{J}$ which is the solution computed in Step 2 on $\mathcal{J} \cap \Omega$, and $\tilde{U} = 0$ on $\mathcal{J} \cap (\mathcal{R} - \Omega)$. We now approximate the discrete Laplacian Δ_h of U on the grid $\mathcal{S}$. For Δ_h we can use the familiar five-point Laplacian, but it is almost as convenient to use the nine-point Laplacian Δ_h^9 (e.g., see [10, section 7.7.2]). Both operators have the form

$$\Delta_h w_{ij} = h^{-2} \sum_{s,t=-1}^{1} a_{st} w_{i+s,j+t}, \tag{4.8}$$

where for Δ_h^5, $a_{st} = -4, 1, 0$ for $|s|+|t| = 0, 1, 2$, respectively, and for Δ_h^9, $a_{st} = -10/3$, $2/3$, $1/6$ for $|s| + |t| = 0, 1, 2$. For a smooth function w, $\Delta_h^5 w = \Delta w + O(h^2)$. For Δ_h^9,

$$\Delta_h^9 w(x, y) = \Delta w + (h^2/12)\Delta^2 w + (h^4/360)(\Delta^3 w + 2D_{xy}^2 \Delta w) + O(h^6) \tag{4.9}$$

(see [5], p. 88). Thus, for harmonic functions, $\Delta_h^9 w - \Delta w = O(h^6)$. Choosing one of these Δ_h, we form an approximation to $\Delta_h U$ by setting

$$F = \Delta_h^9 \tilde{U} \quad \text{on } \mathcal{I}\,, \qquad F = 0 \quad \text{on } \mathcal{S} - \mathcal{I}\,. \tag{4.10}$$

The second case is justified since $\Delta u = 0$ inside Ω, so that $\Delta_h u = O(h^2)$ or $O(h^6)$, and since $U = 0$ outside Ω.

Step 4. Invert the discrete Laplacian. Finally, we compute U^h as the solution of $\Delta_h U^h = F$ on $\mathcal{S}$ with zero boundary condition, using a fast solver. The restriction of U^h to Ω is the desired approximation to the solution u of (4.1). For the case $\Delta_h = \Delta_h^9$, we use the program FFT9, a fast Poisson solver; see [16]. (There is a correction built into the program which is not needed here; e.g., see [10], section 8.6.3.) In our numerical experiments, reported in section 6, we find that the solution computed in Ω is accurate to $O(h^2)$ with either Δ_h^5 or Δ_h^9 for the method described here. However, the accuracy improves if we use Δ_h^9 and extend the set of irregular points to the set $\mathcal{J}$ defined above. We have not proved that these results hold in general. Some analysis of this issue was given in [22, pp. 156–157].

Exterior problems. The method just described can easily be modified to solve a Dirichlet problem exterior to a boundary curve, using the appropriate integral representation. (This was noted by Mayo in [22, p. 149].) The solution can be found on a grid in a rectangular region enclosing the curve, again using the fast Poisson solver. In this case we need to compute values of the solution on the boundary of the rectangular region $\mathcal{R}$, using the double layer integral, in order to specify the boundary

value for the Poisson solver. The trapezoidal rule for the integral is highly accurate since $\partial\mathcal{R}$ is away from $\partial\Omega$. The condition at infinity is built in through the integral representation.

The tangential gradient of G. In further applications, we will need to compute integrals along a curve $\mathcal{C}$ with the tangential, rather than normal, derivative of G. Such an integral has the form

$$v(y) = \int_{\mathcal{C}} G_s(x-y) f(x) ds(x) = -\int_{\mathcal{C}} G(x-y) f_s(x) ds(x), \tag{4.11}$$

where G_s means the arclength derivative of G; the second form shows that v is a single layer potential with density function $-f_s$. In parametrized form, rewriting f as a function of α, we have

$$v(y) = \int_0^{2\pi} x'(\alpha) \cdot \nabla G(x(\alpha) - y) f(\alpha)\, d\alpha. \tag{4.12}$$

The integral can be computed in a manner similar to that for the normal derivative. For y near $\mathcal{C}$, we again write $y = x(\alpha_0) + bn(\alpha_0)$ and rewrite $v(y)$ as

$$v(y) = \int_0^{2\pi} x'(\alpha) \cdot \nabla G(x(\alpha) - y)[f(\alpha) - f(\alpha_0)]\, d\alpha. \tag{4.13}$$

We start by computing the sum

$$S = \sum_{j=1}^{J} x'(\alpha_j) \cdot \nabla G_\delta(x(\alpha_j) - y)\, [f(\alpha_j) - f(\alpha_0)]\; d\alpha \tag{4.14}$$

and again add corrections for the smoothing and quadrature errors. These can be computed as in sections 2 and 3; they are

$$T_1 = (2\pi)^{-1} \delta f_0' \tau_0^{-1} (1 + \tfrac{1}{2}\kappa_0 \eta \delta) \left(\sqrt{\pi} e^{-\eta^2} - \pi |\eta| \operatorname{erfc} |\eta| \right), \tag{4.15}$$

$$T_2 = h f_0' \sum_{n=1}^{\infty} \cos(2n\pi\alpha_0/h) \left(-\frac{\eta\sigma}{2} E^-(\eta, n\pi\sigma) + \frac{\sigma}{\sqrt{\pi}} e^{-\eta^2} e^{-n^2\pi^2\sigma^2} \right). \tag{4.16}$$

In (4.15) κ_0 is the curvature at $x(\alpha_0)$, defined by $x_{ss} = \kappa_0 n(\alpha_0)$, and in (4.16)

$$E^-(\eta, \zeta) = e^{2\eta\zeta} \operatorname{erfc}(\eta + \zeta) - e^{-2\eta\zeta} \operatorname{erfc}(-\eta + \zeta)\,. \tag{4.17}$$

Then, the computed value

$$\tilde{v}(y) = S + T_1 + T_2 \tag{4.18}$$

has the same accuracy as obtained previously for the double layer potential.

Gradients of potentials. The gradient of a single or double layer potential can be found by a procedure very similar to that for the Dirichlet problem outlined above, since the derivatives of the potential are harmonic functions. (Again this was noted by Mayo; cf. [21, pp. 293–294]). Suppose a double layer potential $u(y)$ is defined by (4.2) or (4.3) with f known. Then, $\nabla u(y_1, y_2)$ is given by

$$\frac{\partial u}{\partial y_1} = \mathcal{N}\left[f_s x_{1,s}\right] - \mathcal{T}\left[f_s x_{2,s}\right], \quad \frac{\partial u}{\partial y_2} = \mathcal{N}\left[f_s x_{2,s}\right] + \mathcal{T}\left[f_s x_{1,s}\right], \tag{4.19}$$

where we have introduced the notation

$$\mathcal{N}[\mu] = \int_{\mathcal{C}} (\partial_n G)\mu \, ds\,, \quad \mathcal{T}[\nu] = \int_{\mathcal{C}} (\partial_s G)\nu \, ds. \tag{4.20}$$

Each of the four integrals can be computed accurately, since each is in one of the two forms we have treated. On the other hand, if $v(y)$ is a single layer potential defined by

$$v(y) = \int_{\mathcal{C}} G(x-y)\, g(x)\, ds(x), \tag{4.21}$$

then $\nabla v(y_1, y_2)$ is given by the very similar expression

$$\frac{\partial v}{\partial y_1} = -\mathcal{N}\left[g x_{2,s}\right] - \mathcal{T}\left[g x_{1,s}\right]\,, \quad \frac{\partial v}{\partial y_2} = \mathcal{N}\left[g x_{1,s}\right] - \mathcal{T}\left[g x_{2,s}\right]. \tag{4.22}$$

(In fact v in (4.22) is the harmonic conjugate of u in (4.2) if $g = f_s$.)

5. Pressure gradients due to boundaries in fluids. In computing time-dependent, incompressible fluid flow it is often necessary to compute the pressure gradient. When a moving boundary or interface is present in the fluid, one term in the pressure may be written as a harmonic function with boundary condition determined by forces at the interface. The method introduced here might then be useful for computing the gradient of this pressure term at grid points in the fluid. The boundary might separate two different fluids (e.g., see [6]) or it might be an elastic membrane in one fluid [26, 27].

The Navier–Stokes equations of viscous incompressible flow have the form

$$\rho(v_t + v \cdot \nabla v) + \nabla p = \nu \Delta v + F, \tag{5.1}$$

$$\nabla \cdot v = 0, \tag{5.2}$$

where v is the velocity field, ρ the density, p the pressure, ν the viscosity, and F is a force other than that due to pressure and viscous stress. When solutions are computed using the projection method, a preliminary velocity v^* is found at each time step and projected onto the space of divergence-free vector fields. Finding this projection amounts to subtracting a term proportional to ∇p, subject to boundary conditions (cf. [26, 27]). If ρ is constant on each side of an interface, we can obtain Δp away from the interface by taking the divergence of the first equation. The pressure is then determined by this Poisson equation and boundary conditions. Since the boundary value problem for p is linear, we can write p as a sum of terms, with one accounting for a prescribed jump in pressure at the interface (cf. [9]). This term will be harmonic in the interior and have continuous normal derivative at the interface. It can, therefore, be written as a double layer potential. For an interface between two viscous incompressible fluids, with differing density and viscosity, the jump in pressure is determined by the condition that the jump in the normal stress is normal and proportional to the force of surface tension (see [19, 6]). If the fluid occupies $\mathbf{R}^2$ and the interface is a closed curve $\mathcal{C}$, and if there is no viscosity, the pressure term is prescribed by the problem

$$\Delta\phi = 0 \text{ in } \mathbf{R}^2 - \mathcal{C}\,, \quad \phi_+ - \phi_- = \sigma\kappa \text{ on } \mathcal{C}\,, \quad \phi_{n+} - \phi_{n-} = 0 \text{ on } \mathcal{C} \tag{5.3}$$

with a decay condition at infinity, where κ is the curvature at the interface and σ is the coefficient of surface tension, assumed constant (cf. [6]). With viscosity, the jump

in ϕ has a further term involving ∇v. Periodic boundary conditions could be imposed in the method proposed here; see below.

To be specific, we discuss further the case of an elastic membrane in a viscous fluid, the case for which Peskin's immersed boundary method [26, 27] is designed. Assume for now that the fluid occupies $\mathbf{R}^2$. We return to (5.1), (5.2) and assume that the force F is due to a membrane located on the curve $\mathcal{C}$. That is, F will have the form

$$F = (f^{(n)}n + f^{(t)}t)\delta_{\mathcal{C}}, \tag{5.4}$$

where $f^{(n)}$, $f^{(t)}$ are scalar-valued functions on the curve, n is the unit outward normal, and t is the unit tangent to the curve. Here $\delta_{\mathcal{C}}$ is the delta function on the curve $\mathcal{C}$. We use the notation $\langle \cdot, \cdot \rangle$ to indicate the action of a distribution or generalized function on a test function, i.e., a smooth function on $\mathbf{R}^2$ of bounded support; this action generalizes the integral of a product of functions on $\mathbf{R}^2$. Thus, the action of $\delta_{\mathcal{C}}$ on a test function ψ is

$$\langle \delta_{\mathcal{C}}, \psi \rangle = \int_{\mathcal{C}} \psi \, ds. \tag{5.5}$$

The contribution to the pressure p from the membrane force F (assuming constant density) is the distributional solution of

$$\Delta p^{(M)} = \nabla \cdot F. \tag{5.6}$$

We show that the solution of (5.6) with F concentrated on $\mathcal{C}$ can be written in terms of single and double layer potentials on $\mathcal{C}$. We separate $p^{(M)}$ into two terms corresponding to those in F,

$$\Delta p^{(n)} = \nabla \cdot (f^{(n)}\delta_{\mathcal{C}} n), \quad \Delta p^{(t)} = \nabla \cdot (f^{(t)}\delta_{\mathcal{C}} t). \tag{5.7}$$

We verify that

$$p^{(n)}(y) = -\int_{\mathcal{C}} \partial_n G(x-y) f^{(n)}(x) \, ds(x). \tag{5.8}$$

From the definition of distributional derivatives, we have

$$\langle \nabla \cdot (f^{(n)}\delta_{\mathcal{C}} n), \psi \rangle = -\langle f^{(n)}\delta_{\mathcal{C}} n, \nabla \psi \rangle = -\int_{\mathcal{C}} f^{(n)} n \cdot \nabla \psi \, ds = -\int_{\mathcal{C}} f^{(n)} \partial_n \psi \, ds. \tag{5.9}$$

On the other hand, with $p^{(n)}$ the function on $\mathbf{R}^2$ defined by (5.8) we find (again starting with the definition of distributional derivative)

$$\begin{aligned} \langle \Delta p^{(n)}, \psi \rangle = \langle p^{(n)}, \Delta \psi \rangle &= \int_{\Omega \cup (\mathbf{R}-\Omega)} p^{(n)} \Delta \psi \\ &= \int_{\mathcal{C}} \left([p^{(n)}] \partial_n \psi - [\partial_n p^{(n)}] \psi \right) ds = -\int_{\mathcal{C}} f^{(n)} \partial_n \psi \, ds. \end{aligned} \tag{5.10}$$

Here $[p^{(n)}]$ is the boundary value from inside minus that from outside, which from the properties of the double layer potential is $-f^{(n)}$. We have used Green's identity and

the fact that $\Delta p^{(n)} = 0$ on each side of $\mathcal{C}$. Since (5.9) and (5.10) agree with arbitrary ψ, we have verified that $p^{(n)}$, as defined by (5.8), satisfies the equation in (5.7); it is the unique solution with the appropriate behavior at infinity. In an entirely similar way we can show that

$$p^{(t)}(y) = -\int_{\mathcal{C}} \partial_s G(x-y) f^{(t)}(x)\, ds(x)\,. \tag{5.11}$$

Combining this result with the formulas (4.19), (4.22) for gradients of potentials, we can express $\nabla p^{(n)}(y_1, y_2)$ and $\nabla p^{(t)}(y_1, y_2)$ as

(5.12)
$$\frac{\partial p^{(n)}}{\partial y_1} = \mathcal{T}\left[f_s^{(n)} x_{2,s}\right] - \mathcal{N}\left[f_s^{(n)} x_{1,s}\right], \quad \frac{\partial p^{(n)}}{\partial y_2} = -\mathcal{N}\left[f_s^{(n)} x_{2,s}\right] - \mathcal{T}\left[f_s^{(n)} x_{1,s}\right],$$

(5.13)
$$\frac{\partial p^{(t)}}{\partial y_1} = -\mathcal{N}\left[f_s^{(t)} x_{2,s}\right] - \mathcal{T}\left[f_s^{(t)} x_{1,s}\right], \quad \frac{\partial p^{(t)}}{\partial y_2} = \mathcal{N}\left[f_s^{(t)} x_{1,s}\right] - \mathcal{T}\left[f_s^{(t)} x_{2,s}\right].$$

Now suppose that the curve $\mathcal{C}$ is contained in a region covered by a rectangular grid. It follows from the representation above that each term in $\nabla p^{(n)}, \nabla p^{(t)}$ can be computed at the grid points using the method of section 4. In this case we need to compute Δ_h at irregular points on both sides of $\mathcal{C}$. We also need to compute values at the boundary of the grid to furnish the boundary condition for the Poisson solver; these values can be computed routinely since they are given by nonsingular integrals.

A simple modification will allow us to compute pressure gradients with periodic boundary conditions, rather than the free space condition of the layer potentials. Let w^{free} be one of the pressure gradient terms above, and let w^{per} be the corresponding term with periodic boundary condition. Then $\Delta(w^{per} - w^{free}) = 0$ across $\mathcal{C}$, so that $w^{per} - w^{free}$ is smooth across $\mathcal{C}$, even though the separate terms are not. Consequently, $\Delta_h(w^{per} - w^{free})$ is high order accurate in h. To compute w^{per} at the grid points, we compute $\Delta_h w^{free}$ at the irregular points as before, and then solve the Poisson equation with periodic boundary conditions to obtain w^{per} to high accuracy. If we need values of w^{per} on the curve $\mathcal{C}$, we can first compute w^{free} directly from (5.12), (5.13) and then add the value of $w^{per} - w^{free}$ interpolated from the grid; the latter will be accurate since $w^{per} - w^{free}$ is smooth near $\mathcal{C}$. A similar strategy for computing a periodic solution using values of the free space solution was used in [9].

6. Computational examples. First we illustrate the method of integration in the simplest case, a double layer potential with one mode on a circle. We take $\mathcal{C}$ to be the unit circle, parametrized by $\alpha = \theta$, the angle in polar coordinates. We choose $f(\theta) = 2\sin 3\theta$; then the potential u of (1.3) is $u(r,\theta) = r^3 \sin 3\theta$ for $r < 1$ and $-r^{-3}\sin 3\theta$ for $r > 1$. We use the method described in section 1 to calculate $u(r,\theta)$ for $\theta = .7$ and r slightly greater than 1. Then, u is close to $-\sin 2.1 \approx -.86$. (Results for $r < 1$ are similar.) Table 1 shows the errors and correction terms (1.9), (1.11) for various choices of M, the number of points on the circle, and $\rho = \delta/h$, with $r - 1$ about $h/3$. Here *Raw Err* means the error $S - u$ in approximating u by the sum S of (1.7) without corrections. *Corr Err* means the error after the corrections, i.e., $(S + T_1 + T_2) - u$. We include the case without smoothing, labeled as $\delta/h = 0$.

In each case the corrected error is smaller than the raw error, and in most cases considerably so. With $M = 800$ the errors in the corrected values are all smaller by at least a factor of 100 than that for the sum S without smoothing. It is evident that the

TABLE 1
The double layer potential on a circle.

M	$r-1$	δ/h	Raw Err	$T1$	$T2$	Corr Err
200	.01	.0	-3.73E-3	0.00E00	3.75E-3	1.68E-5
		.2	-3.65E-3	1.46E-6	3.67E-3	1.41E-5
		.5	-6.80E-4	8.66E-5	5.72E-4	-2.18E-5
		1.0	-3.50E-4	3.69E-4	2.40E-7	1.88E-5
		1.5	-6.29E-4	6.90E-4	7.56E-13	6.13E-5
		2.0	-8.82E-4	1.02E-3	1.84E-20	1.42E-4
		3.0	-1.24E-3	1.70E-3	4.63E-42	4.65E-4
400	.005	.0	6.50E-4	0.00E00	-6.27E-4	2.28E-5
		.2	6.47E-4	3.65E-7	-6.25E-4	2.32E-5
		.5	1.05E-4	2.17E-5	-1.13E-4	1.37E-5
		1.0	-8.97E-5	9.22E-5	-4.75E-8	2.41E-6
		1.5	-1.65E-4	1.73E-4	-1.50E-13	7.69E-6
		2.0	-2.38E-4	2.56E-4	-3.64E-21	1.78E-5
		3.0	-3.66E-4	4.25E-4	-9.16E-43	5.86E-5
800	.0025	.0	-8.90E-4	0.00E00	8.84E-4	-5.99E-6
		.2	-8.21E-4	9.11E-8	8.15E-4	-6.18E-6
		.5	-1.13E-4	5.41E-6	1.04E-4	-3.36E-6
		1.0	-2.28E-5	2.30E-5	4.38E-8	2.87E-7
		1.5	-4.22E-5	4.32E-5	1.38E-13	9.63E-7
		2.0	-6.17E-5	6.40E-5	3.35E-21	2.23E-6
		3.0	-9.89E-5	1.06E-4	8.44E-43	7.34E-6

TABLE 2
The double layer potential on an ellipse.

M	*Dist*	δ/h	Raw Err	$T1$	$T2$	Corr Err
200	.01	.0	7.29E-3	0.00E00	-7.43E-3	-1.37E-4
		.5	3.50E-3	-1.33E-4	-3.49E-3	-1.24E-4
		1.0	6.48E-4	-5.65E-4	-9.19E-5	-9.35E-6
		2.0	1.67E-3	-1.57E-3	-9.77E-11	1.02E-4
		3.0	2.94E-3	-2.61E-3	-1.90E-20	3.39E-4
800	.0025	.0	2.28E-3	0.00E00	-2.27E-3	1.16E-5
		.5	6.80E-4	-8.30E-6	-6.60E-4	1.16E-5
		1.0	5.32E-5	-3.53E-5	-1.68E-5	1.09E-6
		2.0	9.96E-5	-9.80E-5	-1.78E-11	1.56E-6
		3.0	1.68E-4	-1.63E-4	-3.47E-21	5.10E-6

two corrections are complementary: For small δ/h, the error is mostly canceled by the quadrature correction T_2, and T_1 is insignificant, whereas for $\delta/h \geq 1$, the smoothing correction T_1 mostly compensates for the error and T_2 is negligible. Here $\tau = 1$, so that $\sigma = \rho$, and the behavior for $\sigma \geq 2$ is consistent with the remarks in section 1. With $\delta/h = 2$, the error for each M is $.57h^3$, confirming the expectation from section 1 that the error at a fixed angle has the form δ^3 times a function of $(r-1)/\delta$.

Since the circle is a special case, we present a similar example where the curve is the ellipse $(x/2)^2 + y^2 = 1$. We parametrize the ellipse as $x = 2\cos\alpha, y = \sin\alpha$. We choose f in (1.3) so that the double layer potential u is Im $(x+iy)^3 = 3x^2y - y^3$ inside; f has the form $A\sin\alpha + B\sin 3\alpha$, with coefficients A, B determined by matching formulas for u inside and out. (We are grateful to Gregory Baker for pointing out this class of examples.) In Table 2 we report results where the point of evaluation is inside the ellipse, at distance *Dist* along the normal line through the point with $\alpha = .7$. Here $u \approx 4.25$ and $\tau \approx 1.5$. The results are similar to those for the circle. Again with $M = 800$ all corrected errors are better by more than a factor of 100 than for the

TABLE 3
The single layer potential on an ellipse.

M	$Dist$	δ/h	Raw Err	$T1$	$T2$	Corr Err
200	.01	.0	-3.76E-3	0.00E00	3.49E-3	-2.72E-4
		.5	1.42E-3	-3.78E-3	1.83E-3	-5.27E-4
		1.0	1.57E-2	-1.61E-2	1.34E-4	-2.57E-4
		2.0	4.43E-2	-4.46E-2	5.33E-10	-3.13E-4
		3.0	7.30E-2	-7.41E-2	2.31E-19	-1.09E-3
400	.005	.0	-3.37E-3	0.00E00	3.39E-3	2.09E-5
		.5	-1.67E-3	-1.89E-3	3.61E-3	5.25E-5
		1.0	7.74E-3	-8.03E-3	3.13E-4	2.29E-5
		2.0	2.22E-2	-2.23E-2	1.24E-9	-3.88E-5
		3.0	3.69E-2	-3.70E-2	5.37E-19	-1.36E-4
800	.0025	.0	1.36E-3	0.00E00	-1.38E-3	-2.21E-5
		.5	2.32E-3	-9.42E-4	-1.40E-3	-2.96E-5
		1.0	4.12E-3	-4.01E-3	-1.19E-4	-1.17E-5
		2.0	1.11E-2	-1.11E-2	-4.71E-10	-4.84E-6
		3.0	1.85E-2	-1.85E-2	-2.04E-19	-1.69E-5

TABLE 4
A pressure gradient outside a circle.

M	$r-1$	δ/h	Raw Err 1	Corr Err 1	Raw Err 2	Corr Err 2
200	.01	.0	1.93E-2	-1.75E-4	-7.18E-3	-7.35E-5
		1.0	-4.68E-2	4.06E-5	-5.71E-2	4.85E-5
		2.0	-1.29E-1	4.14E-4	-1.58E-1	5.73E-4
400	.005	.0	3.08E-3	2.41E-5	8.96E-3	-4.58E-5
		1.0	-2.35E-2	5.58E-6	-2.89E-2	6.43E-6
		2.0	-6.51E-2	5.23E-5	-8.02E-2	7.28E-5
800	.0025	.0	1.30E-3	-1.09E-5	-5.73E-3	5.81E-6
		1.0	-1.18E-2	5.72E-7	-1.45E-2	7.28E-7
		2.0	-3.26E-2	6.58E-6	-4.04E-2	9.17E-6

sum without smoothing. The next example illustrates the corrections (4.15)–(4.17) for the single layer potential. We use a test problem similar to the last one, with the same ellipse. We choose $f(\alpha)$ in the same form, with coefficients so that the single layer potential v of (4.11) is Re $(x+iy)^3 = x^3 - 3xy^2$ inside the ellipse. We calculate values of v at the same locations as before, using (4.18). The value of v is about 1.67. The results are displayed in Table 3. They are qualitatively similar to the results for the double layer potential. However, the smoothing correction T_1 is larger, reflecting the fact that it is $O(\delta)$ in the present case but $O(\delta^2)$ in the double layer case. The corrected error with larger values of δ/h is $O(h^3)$.

Next we illustrate the pressure gradient of section 5. With $\mathcal{C}$ the unit circle and $f(\theta) = -2\sin 3\theta$, let p be the solution of $\Delta p = \nabla\cdot(f\delta_{\mathcal{C}} n)$, as in (5.7). That is, $\Delta p = 0$ away from $\mathcal{C}$; $p_+ - p_- = f$ on $\mathcal{C}$; and $\partial_n p$ is continuous across $\mathcal{C}$. The exact solution, as in the first example, is $p = r^3 \sin 3\theta$ for $r < 1$ and $p = -r^{-3}\sin 3\theta$ for $r > 1$. We calculate ∇p using (5.12) for $\theta = .7$ and r slightly greater than 1. The integrals of (4.20) are calculated as in the previous examples, with corrections as in (1.9), (1.11), (4.15), (4.16). The exact value of ∇p is close to $(1.00, 2.83)$. The results are shown in Table 4; raw and corrected errors are shown for each component of $\nabla p = (\partial_1 p, \partial_2 p)$. In each case the corrected value has a much smaller error. Once again we see $O(h^3)$ convergence with $\delta/h = 2$.

Our final example is the solution of a Dirichlet problem using the method of section 4. We take the domain to be the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a =$

TABLE 5
Errors in the Dirichlet problem.

	Max Err at Bdry		Case I		Case II		Case III	
N	Raw	Corr	Max Err	L^2 Err	Max Err	L^2 Err	Max Err	L^2 Err
32	2.96E-3	3.10E-4	8.84E-4	4.84E-4	6.01E-4	1.44E-4	3.14E-4	5.07E-5
64	6.99E-4	4.10E-5	2.29E-4	1.24E-4	1.27E-4	2.86E-5	4.80E-5	5.22E-6
128	1.70E-4	5.03E-6	5.89E-5	3.18E-5	2.90E-5	6.52E-6	5.92E-6	5.31E-7
256	4.36E-5	6.27E-7	1.50E-5	8.08E-6	6.52E-6	1.46E-6	1.26E-6	1.09E-7

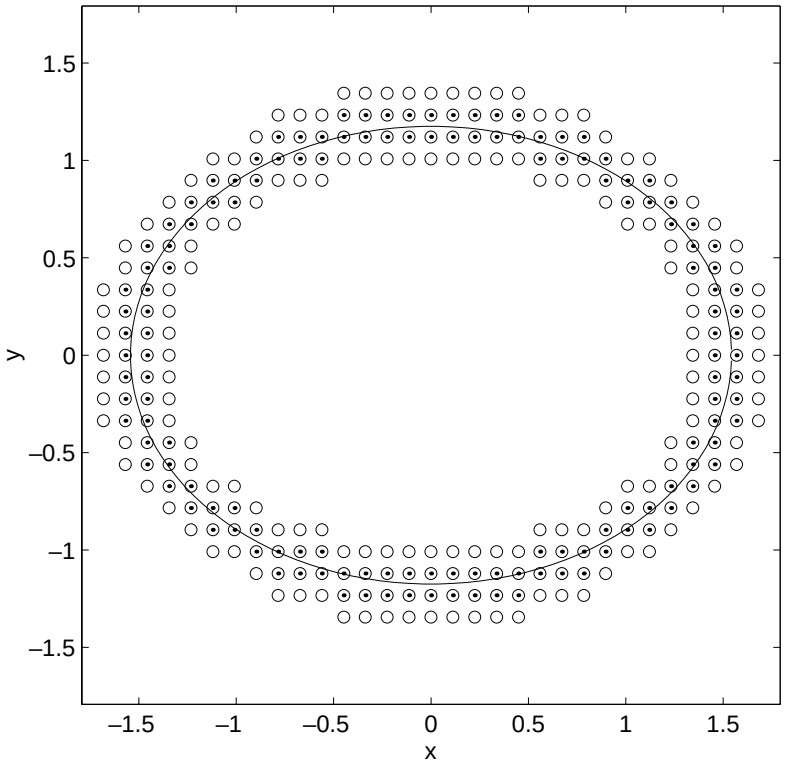

FIG. 1. *Irregular points and their stencils.*

$\cosh(1)$, $b = \sinh(1)$, embedded in a computational square with side $2a + .5$. We prescribe the boundary value on the ellipse to be the harmonic function $\exp((x + \sqrt{3}y)/2)\cos((-\sqrt{3}x + y)/2)$. The computed solution will be compared with this exact solution in the interior. We introduce a grid in the square with N points on each side. With N chosen, we use $M = 4N$ points to discretize the ellipse, parametrized as $x = a\cos\alpha, y = b\sin\alpha$, with $\alpha_i = (i + \frac{1}{2})h$, $h = 2\pi/M$. (The sign of T_2 must be reversed, as mentioned in section 1.) We present results in Table 5 with three variants of the method of section 4: Case I is the method as described with the five-point Laplacian; Case II is the same but with the nine-point Laplacian. Case III also uses the nine-point Laplacian, but the set of irregular points, on which Δ_h^9 is computed using the integral representation, is extended to the set $\mathcal{J}$ of section 4, consisting of all points which occur in any stencil crossing the ellipse. In all cases, the integral (1.3) must be computed at each point inside the ellipse which occurs in the stencil of some irregular point. Figure 1 shows this set of points, with the irregular points marked

by dotted circles, in the case of the five-point Laplacian with $N = 32$. That is, the points shown are the sets $\mathcal{I}$ and $\mathcal{J}$ of section 4 in the five-point case.

In the first two columns of Table 5 we give the maximum errors in the solution, before and after the corrections T_1, T_2, for the set of points for which the solution is computed from the integral. This set of points depends on N, but for each N the maximum error was found to be independent of the variant I, II, or III. We chose $\delta = 2h$, so that the smoothing error should be dominant in computing the integral. The maximum corrected error is clearly $O(N^{-3})$, confirming this expectation.

For each of the three variants of the Dirichlet problem, we display the maximum error and L^2 error, on the set of grid points inside the ellipse, for the solution obtained from the Poisson solver. In all cases we see convergence as N increases, to about $O(N^{-2})$ or better. In Case I, with five-point Laplacian, the order of both errors is slightly worse than $O(N^{-2})$. However, the maximum error is considerably larger than that for the smaller set of values computed from the integral. In Case II, with the nine-point Laplacian, the errors are improved somewhat, especially in the L^2 norm, and the order of convergence is better than $O(N^{-2})$, but we still see a larger maximum error on the grid. In Case III, with the nine-point Laplacian and the extended set of irregular points, the maximum error on the grid is now comparable to that computed from the integrals. The errors are improved considerably. The convergence appears to be about $O(N^{-3})$ up to $N = 128$, but it is closer to $O(N^{-2})$ when we reach $N = 256$. If we repeat the last case with $\delta/h = 1$ rather than 2 (not displayed) we find slightly smaller errors with convergence about $O(N^{-2})$. The accurate solutions obtained demonstrate the feasibility of this method for solving elliptic boundary value problems.

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