A note on finite difference discretizations for Poisson equation on a disk

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A simple second-order finite difference treatment of polar coordinate singularity for Poisson equation on a disk is presented. By manipulating the grid point locations, we can successfully avoid finding numerical boundary condition at the origin so that the resulting matrix is simpler than traditional schemes. The treatments for Dirichlet and Neumann boundary problems are slightly different by adjusting the radial mesh width. Furthermore, the present discretization can be applied with slight modifications to Helmholtz-type equations. The numerical evidence shows that the second-order accuracy can also be preserved. © ??? John Wiley & Sons, Inc.

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I. INTRODUCTION

Many physical problems involve solving elliptic equations on polar or cylindrical domains. The first step is to transform the rectangular coordinate system into the convenient polar or cylindrical coordinates. Thus, we can rewrite the governing equations in those new coordinates. Let us consider Poisson equation on a unit disk $\Omega = \{(x, y) : x^2 + y^2 < 1\}$,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F \quad \text{in } \Omega, \tag{1.1}$$

with the Dirichlet $U = G$, or the Neumann $\frac{\partial U}{\partial \mathbf{n}} = G$ boundary conditions on $\partial\Omega$.

Applying the polar coordinate transformation, $x = r\cos\theta, y = r\sin\theta$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$, and setting $u(r, \theta) = U(r\cos\theta, r\sin\theta)$, $f(r, \theta) = F(r\cos\theta, r\sin\theta)$, and $g(\theta) = G(\cos\theta, \sin\theta)$, then Eq. (1.1) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f(r, \theta) \quad 0 < r < 1, \, 0 \le \theta < 2\pi, \tag{1.2}$$

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with $u(1,\theta) = g(\theta)$ (Dirichlet), or $\frac{\partial u}{\partial r}(1,\theta) = g(\theta)$ (Neumann).

Eq. (1.2) has an apparent singularity at the origin $r = 0$. It is important to realize that the cause of singularity is due to the representation of the governing equation in polar coordinate system. Thus, the solution itself is no way singular at the origin if f is smooth enough. In order to have the desired regularity and accuracy, the traditional finite difference scheme uses a uniformly integered grid with some condition at the origin [4]. This pole condition acts as a numerical boundary condition at the origin which is needed in finite difference scheme. However, from the rectangular coordinate point of view, there is no need to impose any conditions.

There is another simple approach to solve Eq. (1.2) described in [3]. This method uses the truncated Fourier series expansions to derive a singular ordinary differential equation for each Fourier coefficient, then solves the singular ODE by the centered difference method. The total operations count is roughly the same as our present scheme. However, the natural boundary condition at the origin must be derived and used as a numerical boundary value in the finite difference setting.

In the following section, we will present a finite difference discretization for Eq. (1.2) which is second-order accurate without imposing any pole conditions.

II. FINITE DIFFERENCE DISCRETIZATION

Let us first consider the Dirichlet boundary problem. We choose a grid which the grid points are half-integered in radial direction and integered in azimuthal direction, that is,

$$r_i = (i - \frac{1}{2})\,\Delta r, \qquad \theta_j = (j-1)\,\Delta\theta \tag{2.1}$$

where $\Delta r = \frac{2}{2N+1}$, $\Delta\theta = \frac{2\pi}{M}$, and $i = 1, 2, \ldots, N+1; j = 1, 2, \ldots, M+1$. Note that, by the choice of the radial mesh width, the boundary values are defined on the grid points. Let the discrete values be denoted by $u_{ij} \approx u(r_i, \theta_j)$, $f_{ij} \approx f(r_i, \theta_j)$, and $g_j = g(\theta_j)$. Using the centered difference method to discretize Eq. (1.2), for $i = 2, 3, \ldots, N, j = 1, 2, \ldots, M$ we have

$$\frac{u_{i+1,j} - 2\,u_{ij} + u_{i-1,j}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1,j} - u_{i-1,j}}{2\,\Delta r} + \frac{1}{r_i^2}\frac{u_{i,j+1} - 2\,u_{ij} + u_{i,j-1}}{(\Delta\theta)^2} = f_{ij}. \tag{2.2}$$

Among the above representations, the boundary values are given by $u_{N+1,j} = g_j$, and $u_{i0} = u_{i,M}, u_{i1} = u_{i,M+1}$ since u is 2π periodic in θ.

At $i = 1$, we have

$$\frac{u_{2j} - 2\,u_{1j} + u_{0j}}{(\Delta r)^2} + \frac{1}{r_1}\frac{u_{2j} - u_{0j}}{2\,\Delta r} + \frac{1}{r_1^2}\frac{u_{1,j+1} - 2\,u_{1j} + u_{1,j-1}}{(\Delta\theta)^2} = f_{1j}. \tag{2.3}$$

Since $r_1 = \frac{\Delta r}{2}$, we immediately observe that the coefficient of u_{0j} in Eq. (2.3) is zero. It turns out that the scheme does not need any extrapolation for u_{0j} so that there is no pole condition needed.

Let us order the unknowns u_{ij} by first grouping the same ray then moving counter-clockwise to cover the whole domain. Thus, the unknown vector v is defined by

$$v = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}, \qquad u_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{Nj} \end{bmatrix}. \tag{2.4}$$

The remaining problem is to solve a large sparse linear system $Av = b$, where A can be written as ,

$$A = \begin{bmatrix} T-2D & D & & & & D \\ D & T-2D & D & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & D & T-2D & D \\ D & & & & D & T-2D \end{bmatrix}, \tag{2.5}$$

where $D = diag(\beta_1, \beta_2, \ldots, \beta_N)$ with $\beta_i = \frac{1}{(i-1/2)^2(\Delta\theta)^2}, 1 \le i \le N$, and

$$T = \begin{bmatrix} -2 & 1+\lambda_1 & & & & \\ 1-\lambda_2 & -2 & 1+\lambda_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1-\lambda_{N-1} & -2 & 1+\lambda_{N-1} \\ & & & & 1-\lambda_N & -2 \end{bmatrix}, \tag{2.6}$$

with $\lambda_i = \frac{1}{2(i-1/2)}, 1 \le i \le N$. The known vector b is defined by

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}, \qquad b_j = \begin{bmatrix} (\Delta r)^2 f_{1j} \\ \vdots \\ (\Delta r)^2 f_{N-1,j} \\ (\Delta r)^2 f_{Nj} - (1+\lambda_N) g_j \end{bmatrix}. \tag{2.7}$$

Scaling the matrix A by the block diagonal matrix with block sub-matrix D^{-1}, the remaining linear equations can be solved by the famous cyclic reduction algorithm in $O(NM \log N)$ arithmetic operations [1, 2, 6]. The matrix from the traditional finite difference scheme on a uniformly integered grid with the pole condition involves finding extra approximation at $r = 0$ which has one more row and column than A; thus, it cannot be solved directly by the Buneman algorithm [4]. Although, our new scheme does not reduce significant amount of operation counts, the matrix of the new scheme has more succinct form than the traditional one. Furthermore, the discretization used here can be applied directly to the Helmholtz-type equation by just adding more terms in the diagonal part of the matrix A. The Helmholtz-type equation has applications from the numerical integration of time-dependent heat equation, reaction-diffusion equations, and fluid equations.

For the Neumann boundary problem, the grid points are located in the same way as the Dirichlet problem. The slight difference is the choice $\Delta r = \frac{1}{N}$. With this choice of radial mesh width, the discrete values of u are defined midway between boundary so that first derivatives can be centered on the grid points. That is, at $r = 1$,

$$\frac{\partial u}{\partial r} \approx \frac{u_{N+1,j} - u_{Nj}}{\Delta r}. \tag{2.8}$$

So the numerical boundary values $u_{N+1,j}$ can be approximated by $u_{Nj} + g_j \Delta r$. Therefore, we only need to modify T by

$$T = \begin{bmatrix} -2 & 1+\lambda_1 & & & & \\ 1-\lambda_2 & -2 & 1+\lambda_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1-\lambda_{N-1} & -2 & 1+\lambda_{N-1} \\ & & & & 1-\lambda_N & -1+\lambda_N \end{bmatrix}, \tag{2.9}$$

and b_j by

$$b_j = \begin{bmatrix} (\Delta r)^2 f_{1j} \\ \vdots \\ (\Delta r)^2 f_{N-1,j} \\ (\Delta r)^2 f_{Nj} - (1+\lambda_N)\, g_j\, \Delta r \end{bmatrix}. \tag{2.10}$$

It is important to note that the grid used for the Neumann problem turns out to be the popular staggered grid used for most of Neumann boundary problems [5].

III. NUMERICAL RESULTS

In this section, we perform two numerical tests for the new finite difference discretization described in the previous section. In the first test, we make a comparison between the numerical results of the present scheme (Scheme 1) and the traditional finite difference scheme on a uniformly integered grid with pole condition treatment (Scheme 2) [4]. Table I shows the maximum errors for three different exact solutions (which have different smoothness at the origin) of Poisson equation on a unit disk. The number N is the number of points used in the radial direction, while 2N points used in the azimuthal direction. We can easily see the present results are comparable with the traditional finite difference results. Furthermore, for the smooth solution $u(r,\theta) = e^{r(\cos\theta + \sin\theta)}$, both schemes show second-order convergence.

In the second test, we apply our new scheme to the following modified Helmholtz equation on a disk,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \left(\frac{1}{r^2} + \alpha\right) u = f(r,\theta). \tag{3.1}$$

Together with Dirichlet boundary conditions, the above equation arises from the projection method of the finite difference solution for intermediate velocity of the incompressible Navier-Stokes equations. We choose three different smooth exact solutions in this test. All three cases, the parameter $\alpha = 1$. Table II shows the maximum errors and the rate of convergence. For the first case, the discrete Laplacian of $u = r^2$ approximates the Laplacian operator exactly so that the error is roughly equal to the machine precision. For the rest of cases, we can see that the present scheme is indeed second-order convergent.

TABLE I. The maximum errors of both schemes for different solutions.

	N=8	N=16	N=32
$u(r,\theta) = e^{r(\cos\theta+\sin\theta)}$			
Scheme 1	1.395E-02	3.524E-03	8.881E-04
Scheme 2	1.363E-02	3.488E-03	8.824E-04
$u(r,\theta) = r^3$			
Scheme 1	1.303E-02	3.562E-03	9.322E-04
Scheme 2	1.097E-02	3.257E-03	8.904E-04
$u(r,\theta) = r^{2.5}$			
Scheme 1	8.421E-03	2.514E-03	7.053E-04
Scheme 2	9.106E-03	2.825E-03	7.980E-04

TABLE II. The convergence analysis.

	N=8	N=16	N=32
$u(r,\theta) = r^2$	9.437E-16	2.720E-15	3.053E-15
$u(r,\theta) = \sin(r\cos\theta)$	1.342E-03	3.447E-04	8.645E-05
rate		1.961	1.995
$u(r,\theta) = e^{r(\cos\theta+\sin\theta)}$	1.138E-02	2.865E-03	7.197E-04
rate		1.990	1.993

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