Currently, my research interest is concerned with analysis and computation of the Partial Differential Equations (PDEs) arising from geophysical fluid dynamics. For example, equations related to weather prediction and oceanography are the inviscid Primitive Equations (PEs) and the Shallow Water Equations (SWEs).
In addition, I am interested in the analysis and numerical methods for the Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs) which give us different points of view to understand the world. It has also been suspected that fluid equations with adding noise perturbation such as the stochastic Navier-Stokes Equations and stochastic Euler Equations might be an important mathematical model for understanding the turbulence of a fluid with a high Reynold number. In geophysical fluid mechanics, stochastic terms are used to model poorly understood phenomena.
The research statement is organized as follows. In Section 1, research projects related to the inviscid 3D Primitive Equations are described. In Section 2, future research projects related to the shallow water equations are described.

1 Research Projects

In this section, research projects related to the inviscid 3D Primitive Equations are described.

1.1 Numerical approximation of the 3D inviscid Primitive Equations in a limited domain

The background of this work is based on a major computational issue for the geophysical fluid dynamics. Limited area models (LAMs) are often used to achieve high resolutions over a region of interest such as regional weather forecasts and simulations of coastal flows and gulf streams. The challenge for using such models is that no physical laws will provide natural boundary conditions at the nonphysical boundaries. Furthermore, we want the lateral boundary conditions to be transparent. The difficulties for lateral boundary conditions are on two sides. On the computational side, if the proposed boundary conditions are not suitable, it is well-known that errors at the boundary will propagate and advect into the modeled domain and have a major impact inside the domain. On the mathematical side, the negative result of Oliger and Sundstrom [19] showed the ill-posedness of the inviscid PEs for any set of local boundary conditions.

In [22], the authors have investigated the inviscid PEs in space dimension two and an infinite set of boundary conditions has been proposed. The well-posedness of the corresponding linearized equations has been established in [23] and numerical simulations have been performed in [24] for the linearized equations and for the full nonlinear equations. Note that the nonoccurrence of blow-up in the latter case supports the (yet unproved) conjecture that the proposed nonlocal boundary conditions are also suitable for the nonlinear equations.
In [6], the authors considered a 2.5D model for the equations with three orthogonal finite elements in the y-direction. The well-posedness of the linearized equations was established in [6], and the numerical simulations of the nonlinear equations on a nested set of domains were discussed in [9].
In [25], the authors obtained an infinite set of nonlocal boundary conditions for the 3D inviscid primitive equations by studying the stationary problem associated with the linearized equations.
In our work [7], [8], we discussed the numerical simulations of the 3D nonlinear inviscid primitive equations on a nested set of domains. The 3D primitive equations, linearized around a uniformly stratified flow (see [23], [24], [6] and [9]), read

$$\begin{cases} u_t + \bar{U}_0 u_x + \phi_x - fv + B(u,v,w;u) = 0, \\ v_t + \bar{U}_0 v_x + \phi_y + fu + B(u,v,w;v) + f\bar{U}_0 = 0, \\ \psi_t + \bar{U}_0 \psi_x + N^2 w + B(u,v,w;\psi) = 0, \\ u_x + v_y + w_z = 0, \\ \phi_z = \psi. \end{cases} \tag{1.1}$$

where u, v and w are the perturbation variables of the three velocity components, ϕ is the perturbation variable of the pressure, ψ is the perturbation variable of the temperature; f is the Coriolis force parameter, N is the Brunt–Väisälä (buoyancy) frequency, assumed to be constant in the current study; $B(u,v,w;\theta) = u\theta_x + v\theta_y + w\theta_z$ for $\theta = u, v$, or ψ.
After performing the normal mode expansions in the vertical direction (see [19] and [29]), we are presented with an infinite set of 2D equations. We supplement the equation for the zero (barotropic) mode with the boundary conditions proposed in [7]. The corresponding equations resemble the Euler equations of incompressible flows with marked differences. We established the well-posedness of the corresponding linearized problems using the linear semi-group approach. One step for the proof is to show the well-posedness of the following unusual boundary value problem of the

Poisson equation:

$$\begin{cases} -\bar{U}_0 \triangle v = F_{1y} - F_{2x} (\in H^{-1}(\mathcal{M}')), \\ v = 0 \text{ at } x = 0,\ y = 0,\ L_2, \\ \int_0^{L_1} v(x,y)\, dx = 0. \end{cases} \tag{1.2}$$

For numerical schemes, due to its resemblance with the classical Navier-Stokes equations and Euler equations, we discretize the barotropic (zeroth mode) equations by the classical projection method (see e.g.[10] and [28]), and by the pressure–correction method (see [11], [15] and [30]) and study their stability.
For the higher modes, i.e. the subcritical and supercritical modes, the proposed boundary conditions are based on the directions of the characteristics at the boundary. For the subcritical modes, two boundary conditions are imposed at the left boundary, and one boundary condition is prescribed at the right boundary in the x-direction. For the supercritical modes, three boundary conditions are prescribed at the left boundary, but no boundary condition is imposed on the right boundary. Note that these proposed boundary conditions are different from those proposed in [25] and [26]. We believe that the well-posedness of the linearized systems corresponding to higher modes and supplemented with the proposed boundary conditions in [8] can be established in the same way as in [25] and [26]. This will appear elsewhere. For numerical schemes for the higher modes, we use the splitting-up method for the discretization and advance the unknowns along the x- and y- directions in separate sub-steps and treat the forcing term explicit.
For each mode, the numerical schemes involve the integral of the nonlinear terms. In this study, we only need to consider a small number (≤ 10) of modes, and it is then appropriate and sufficient to transform these integrals into the sums of suitable Fourier coefficients.
In order to test whether the proposed boundary conditions are transparent, two numerical simulations are performed. An initial simulation is carried out on a large domain with homogeneous boundary conditions. Using the data from the initial simulation as boundary data, we then perform a second simulation of the same equations on the middle-half domain. Then we consider the data from the initial simulation as the true solution and compare these two results over the middle-half domain by computing the relative errors in the L^2 and L^∞ norms. The relative errors for u, v, ψ , and ϕ, in both the L^2 and L^∞ norms, are of the order $O(10^{-2})$, and the relative errors for w are of the order $O(10^{-1})$(Figure 1- Figure 5).

In conclusion, the absence of blow up supports the idea that the proposed boundary conditions are suitable for the nonlinear equations and that the proposed numerical

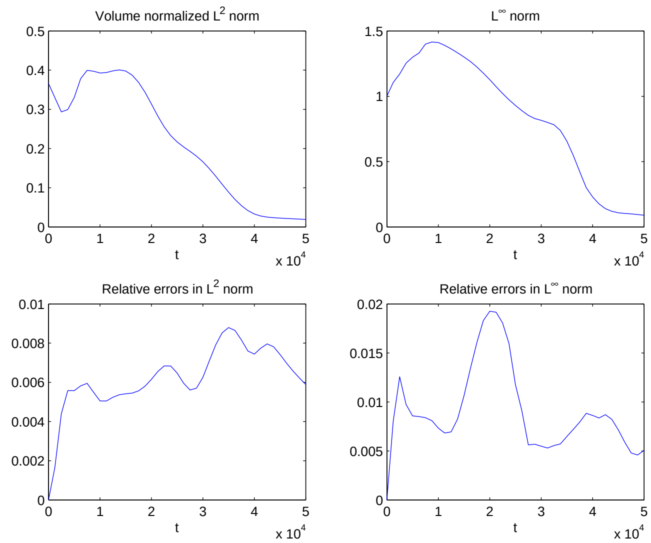

Figure 1: Top row: evolution of the solution u in the L^2 and L^∞ norms. Bottom row: evolution of the relative errors for u in the L^2 and L^∞ norms.

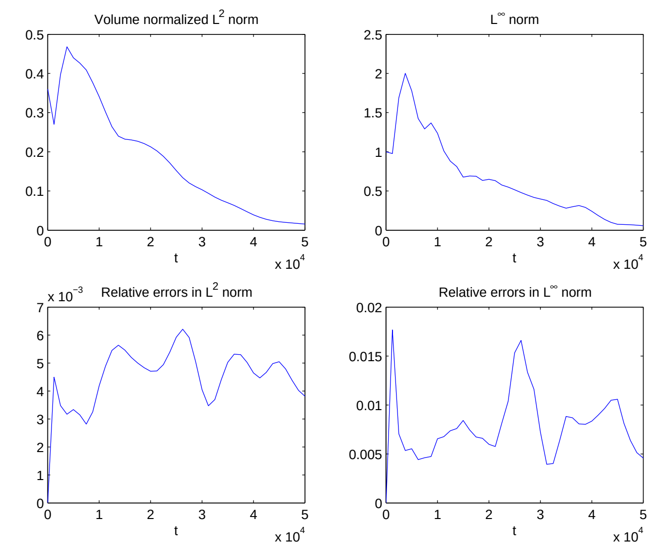

Figure 2: Top row: evolution of the solution v in the L^2 and L^∞ norms. Bottom row: evolution of the relative errors for v in the L^2 and L^∞ norms.

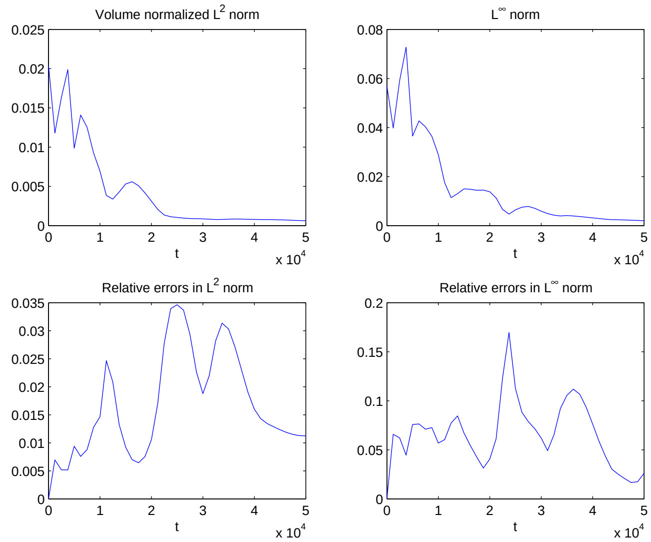

Figure 3: Top row: evolution of the solution w in the L^2 and L^∞ norms. Bottom row: evolution of the relative errors for w in the L^2 and L^∞ norms.

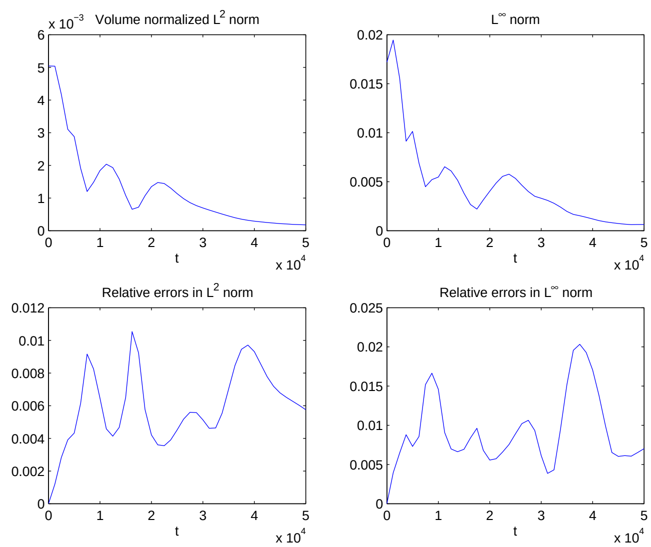

Figure 4: Top row: evolution of the solution ψ in the L^2 and L^∞ norms. Bottom row: evolution of the relative errors for ψ in the L^2 and L^∞ norms.

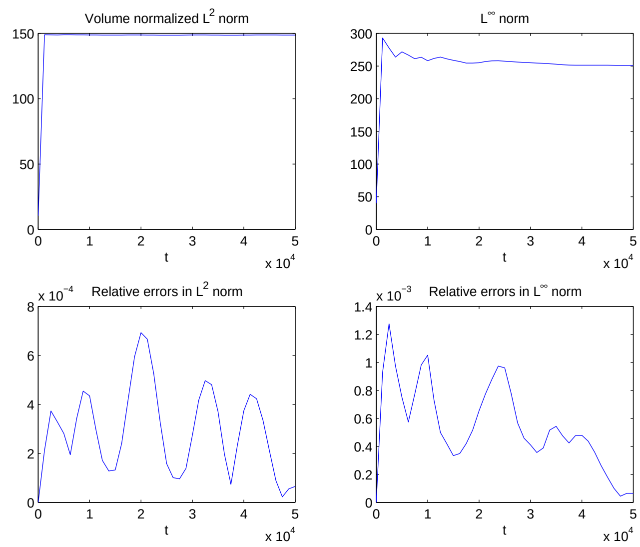

Figure 5: Top row: evolution of the solution ϕ in L^2 and L^∞ norms. Bottom row: evolution of the relative errors for ϕ in L^2 and L^∞ norms.

schemes are stable. The fact that the numerical results match very well on the middle-half domain confirms the transparency property of the proposed boundary conditions.

2 Shallow Water Equations : Current Research and Future Plans

2.1 Analysis and Computation for the Shallow Water Equations with topography

The shallow water equations have been commonly used to describe the evolution of a shallow layer of fluid. In geophysical fluid dynamics, the shallow water equations are considered as a simplification of the primitive equations. We are then usually interested in situations related to turbulence and the height h of the flow does not vanish. We start with the shallow water equations in space dimension one.

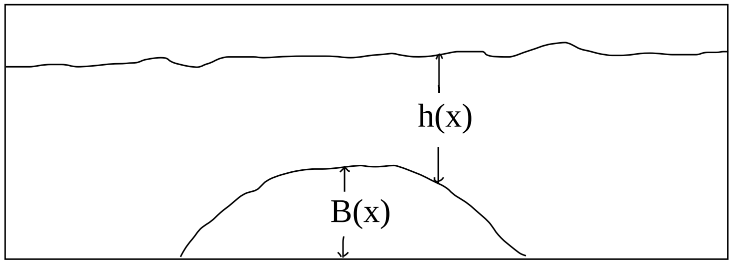

Figure 6: The shallow water model: h is the height of the water level, and B is the bottom profile

The one dimensional shallow water equations read:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + g\dfrac{\partial h}{\partial x} - fv = -g\dfrac{\partial B(x)}{\partial x}, \\ \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + fu = 0, \\ \\ \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + \dfrac{\partial u}{\partial x}h = 0, \end{cases} \tag{2.1}$$

where u and v are the x and y components of the velocity, $B(x)$ is the bottom function of the topography, h is the height of the water, $h+B$ is the water level of the free surface (Figure 6), g is the gravitational acceleration, and f is the Coriolis parameter; f is usually a linear function of y, $f = f_0(1+\beta y)$. Here, we take f conatant for the sake of simplicity, $f = f_0$.
The purpose of this work is to perform numerical simulations using suitable boundary conditions and suitable numerical schemes which guarantee efficiency, accuracy and transparency of the boundary so that the waves freely move in and out of the domain. In our current work [27], two types of characteristic boundary conditions (linear and nonlinear) are considered. Linear characteristic boundary conditions have been applied for a long time, see e.g. [5], [16], [17] and [20]. The nonlinear characteristic boundary conditions that we implemented in [27] are based on the theoretical work of Benzoni and Serre [4] which relates to boundary value problems for linear and nonlinear hyperbolic equations. After our work was completed, we found that these boundary conditions were also proposed in [18] in the context of a numerical work. In [18], the authors added lateral viscosity terms in the momentum equations and

performed numerical simulations of both one-layer and two-layer shallow water models with the proposed linear and nonlinear characteristic open boundary conditions. But, in our work, **the nonviscous shallow water equations** are considered.
The semidiscrete central-upwind method presented in [2],[13] is applied for the space discretization. One of the important features of this method is that it respects the direction of wave propagation by measuring the one-sided local speeds. Furthermore, this method is simple because there are no Rienmann solvers nor characteristic decompositions involved. For the time discretization, the Runge-Kutta method of second order is used, which is one of the standard ODEs' solvers.
Several numerical experiments for subcritical and supercritical flows with these two boundary conditions were performed in [27]. For example, we considered a small perturbation of a steady state subcritical flow which was modified from [14] or [1]. In this example, the small disturbance generates two waves and these two waves propagate at the characteristic speeds $\pm\sqrt{gh}$ and flow out of the domain smoothly (without being reflected at the boundary). The other numerical experiments include transcritical flows and supercritical flows over a hump and a trapezoidal obstacle. The transcritical flows produce a stationary shock. For supercritical flows, the free water surface of the steady states arrives at a large elevation of water above the hump. For subcritical or supercritical flows, the boundary value problems for which the steady states produce a propagating shock (see [3] or [12]) will be studied elsewhere.

2.2 Future Plans

In our future plans, we plan to extend this research project by considering first two dimensional shallow water equations with topography. Namely, we will consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + g\dfrac{\partial h}{\partial x} - fv = -g\dfrac{\partial B(x,y)}{\partial x}, \\[2ex] \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + g\dfrac{\partial h}{\partial y} + fu = -g\dfrac{\partial B(x,y)}{\partial y}, \\[2ex] \dfrac{\partial h}{\partial t} + \dfrac{\partial (uh)}{\partial x} + \dfrac{\partial (vh)}{\partial y} = 0. \end{cases} \tag{2.2}$$

In this two-dimensional case, our goal will be the same as for the one-dimensional case. But considerations for suitable boundary conditions and numerical implementation are different and more complicated due to the additional y direction.

Another important and interesting issue would be to consider a time dependent bottom topography, and more specifically a stochastically defined bottom to account for the roughness of the bottom and incertitudes on the function B. In this case, the bottom function satisfies an SDE of the type:

$$dB = \alpha dt + \beta dW. \tag{2.3}$$

As we said, such a bottom equation would be useful to account for the roughness and the uncertainties in the bottom topography. However, we enter here in the difficult domains of stochastic partial differential equations with white noise in the boundary conditions [21].

We will also consider the boundary value problems for multi-layer shallow water equations. For example, the two-layer shallow water equations in space dimension one read:

$$\begin{cases} \dfrac{\partial h_1}{\partial t} + \dfrac{\partial}{\partial x}(u_1 h_1) = 0, \\[2ex] \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial}{\partial x} u_1 + g \dfrac{\partial}{\partial x}(h_1 + r h_2) = 0, \\[2ex] \dfrac{\partial h_2}{\partial t} + \dfrac{\partial}{\partial x}(u_2 h_2) = 0, \\[2ex] \dfrac{\partial u_2}{\partial t} + u_2 \dfrac{\partial}{\partial x} u_1 + g \dfrac{\partial}{\partial x}(h_1 + h_2) = 0; \end{cases} \tag{2.4}$$

here $r = \dfrac{\rho_2}{\rho_1} < 1$, the ρ_i are the density constants, $i = 1, 2$, and g is the gravitational constant. In this case we will meet a formidable difficulty already pointed out in [19] for the choice of the boundary conditions. Indeed, already in (2.4), we encounter some form of the difficulty shown in [19] and addressed in our other work [8] for the primitive equations, that the boundary conditions can not be of local type.

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